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INTERNATIONAL JOURNAL OF ABSTRACTS
STATISTICAL THEORY AND METHOD

COVERAGE OF JOURNAL

THE aim of this journal of abstracts is to give complete coverage of papers in the field of statistical theory and new contributions to statistical method. Papers which report only applications or examples of existing statistical theory and method will not be included. There are approximately two hundred and fifty journals published in various parts of the world which are wholly or partly devoted to the field of statistical theory and method and which will be brought within the scope of this journal of abstracts. In due course it will be possible to issue a complete list of these journals. In the case of the following journals, however, being those which are wholly devoted to statistical theory—all contributions, whether a paper, note or miscellanea, will be abstracted :

Annals of Mathematical Statistics,
Biometrika
Journal, Royal Statistical Society (Series B)
Bulletin of Mathematical Statistics
Annals, Institute of Statistical Mathematics

Within the larger group of journals, which are not wholly devoted to statistical theory and method, there are some journals which have the vast majority of their contributions in this field. These journals, therefore, will be abstracted on a virtually complete basis :

Biometrics
Metrika
Metron
Review, International Statistical Institute
Technometrics
Saṅkhyā

After experience in the publication of this journal of abstracts it may be found desirable to add further journals to these lists. In any case readers may be assured that all papers properly to be included in this journal of abstracts will be included irrespective of such notification. If any reader of this journal discovers a paper which happens to have been overlooked, the General Editor will be pleased to be informed so that the appropriate abstract can be made : always provided that the date of publication is after 1st October 1958, when the abstracting for this journal commenced.

In addition to the ordinary journals, there are two kinds of publication which fall within the scope of this journal of abstracts. They are the experiment and other research station reports—which occur particularly in the North American region—and the reports of conferences, symposia and seminars. Whilst these latter may be included in the book review sections of journals it is unusual for any individual contribution to be noted at any length. These publications are, in effect, special collections of papers and for this reason the appropriate arrangements will be made for them to be included in this journal. By the same token, abstracts of papers given at conferences and reproduced in an appropriate journal will be disregarded until the definitive publication is available.

FORMAT OF JOURNAL

The abstracts will be up to 400 words long—the recommendation of UNESCO for the " long " abstract service : they will be in the English language although the language of the original paper will be indicated on the abstract together with the name of the abstractor. In addition, the address of the author(s) will be given in sufficient detail to facilitate contact in order to obtain further detail or request an offprint. Suitable indexes will be provided annually in a supplement.

The scheme of classification has been developed upon lines that will facilitate the transfer to punched cards of the code numbers allocated to each abstract: to allow for future development it is suggested that use is made of 4-column fields. Each abstract will have two classification numbers: the primary number in heavy type to indicate the basic topic of the paper and the secondary number in brackets to take account of the most important cross-reference. The sheets of the journal are colour-coded according to the twelve main sections of the classification and it should be noted that it is the main section number of the primary classification which determines the colour code for each abstract. It is believed that this method of colour-coding the pages will provide a distinctive visual aid to the identification of abstracts both when the journal is in bound form or dismantled and filed on cards or in binders. The format and simple binding allows of the following alternative treatments by users of the journal:

(*a*) Leave intact as a shelf-periodical.

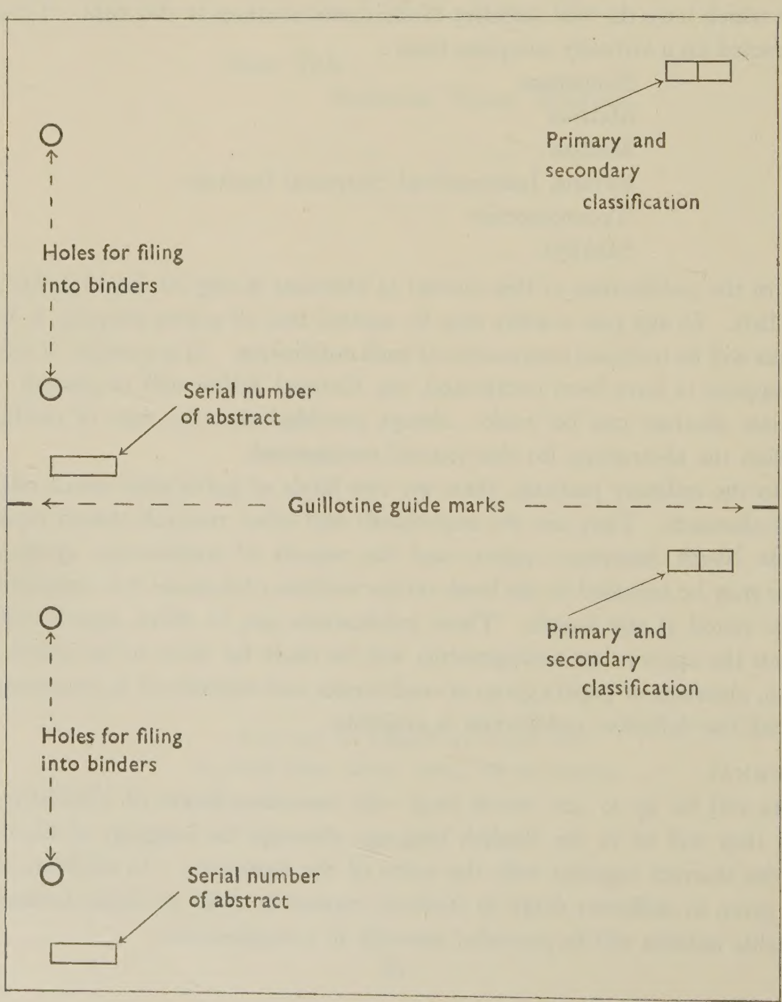
(*b*) Split and filed in page form according to the main sections of the classification.

(*c*) Split and guillotine (single cut) each page ready for:

  (i) pasting onto standard index cards.

  (ii) filing in loose-leaf or other binders—for which the appropriate holes are punched.

It will also be possible for those users who need a completely referenced file to insert skeleton cards or sheets according to the secondary classification number provided on each abstract.



INTERNATIONAL JOURNAL OF ABSTRACTS
STATISTICAL THEORY AND METHOD

LIST OF PUBLICATIONS FOR REGULAR SCANNING

The following list of publications in the statistical and allied fields will be examined regularly for papers which should be abstracted for this Journal. Other journals will also be scrutinised from time to time and a check maintained by scanning other relevant abstracting journals. In addition, abstracts will be included of special collections of papers as published in reports of conferences, symposia and seminars together with the published reports of experiment and other research stations. The coverage of this Journal amounts to at least 400 sources of publication

Acta Acad. Cient., Lima	Acta de la Academia Cientifica, Lima	Peru
Acta Math., Stockh.	Acta Mathematica	Sweden
Acta Math. Acad. Sci. Hung.	Acta Mathematica Academiae Scientiarum Hungaricae	Hungary
Acta Math. Sinica	Acta Mathematica Sinica	China
Acta Pontif. Acad. Sci.	Acta Pontificiae Academiae Scientiarum	Italy
Advanced Management	Advanced Management	U.S.A.
Advanc. Enzymol.	Advances in Enzymology and Related Subjects	U.S.A.
Advanc. Protochem.	Advances in Protochemistry	U.S.A.
Advanc. Vet. Sci.	Advances in Veterinary Science	U.S.A.
Agric. Engng., St Joseph, Mich.	Agricultural Engineering	U.S.A.
Agriculture, Lond.	Agriculture (Min. of Agric.)	Great Britain
Agron. J.	Agronomy Journal	U.S.A.
Allg. Statist. Arch.	Allgemeines Statistisches Archiv	Germany
Amer. Econ. Rev.	American Economic Review	U.S.A.
Amer. Heart J.	American Heart Journal	U.S.A.
Amer. J. Clin. Nutr.	American Journal of Clinical Nutrition	U.S.A.
Amer. J. Clin. Path.	American Journal of Clinical Pathology	U.S.A.
Amer. J. Digestive Diseases	American Journal of Digestive Diseases	U.S.A.
Amer. J. Hum. Genet.	American Journal of Human Genetics	U.S.A.
Amer. J. Hyg.	American Journal of Hygiene	U.S.A.
Amer. J. Med.	American Journal of Medicine	U.S.A.
Amer. J. Physiol.	American Journal of Physiology	U.S.A.
Amer. J. Publ. Health	American Journal of Public Health	U.S.A.
Amer. J. Sci.	American Journal of Science	U.S.A.
Amer. J. Sociol.	American Journal of Sociology	U.S.A.
Amer. J. Vet. Res.	American Journal of Veterinary Research	U.S.A.
Amer. Nat.	American Naturalist	U.S.A.
Amer. Statistician	American Statistician	U.S.A.
Analyt. Chem.	Analytical Chemistry	U.S.A.
An. Soc. Cient. Argent.	Anales de la Sociedad Cientifica Argentina	Argentina
Ann. Acad. Sci. Fenn.	Annales Academiae Scientiarum Fennicae	Finland
Ann. Appl. Biol.	Annals of Applied Biology	Great Britain
Ann. Fac. Econ. Com. Catania	Annali della Facoltà di Economia e Commercio, Catania (University)	Italy
Ann. Fac. Econ. Com. Palermo	Annali della Facoltà di Economia e Commercio, Palermo (University)	Italy
Ann. Fac. Sci. Porto	Annaes da Faculdade de Ciências do Porto	Portugal
Ann. Hum. Genet.	Annals of Human Genetics (London)	Great Britain
Ann. Inst. Poincaré	Annales de l'Institut Henri Poincaré	France
Ann. Inst. Statist. Math., Tokyo	Annals of the Institute of Statistical Mathematics	Japan
Ann. Inst. Sup. Sci. Econ. Financ.	Annaes do Instituto Superior de Ciências Economicas e Financieras	Portugal
Ann. Ist. Statist. Bari	Annali dell'Istituto di Statistica, Bari (University)	Italy
Ann. Math. Statist.	Annals of Mathematical Statistics	U.S.A.
Ann. N.Y. Acad. Sci.	Annals of the New York Academy of Science	U.S.A.
Ann. Sci. Éc. Norm. Sup., Paris	Annales Scièntifiques de l'École Normale Supérieure	France
Ann. Sci. Écon. Appl.	Annales de Sciences Économique Appliquées	France
Ann. Soc. Sci. Bruxelles, I	Annales de la Société Scientifique de Bruxelles—Serie 1	Belgium

Ann. Statist., Rome	Annali di Statistica (Istituto Centrale di Statistica, Rome)	Italy
Ann. Univ. Lyon, A	Annales de l'Université de Lyon—Section A	France
Ann. Univ. Sci. Budapest. Sect. Math.	Annales Universitatis Scientiarum Budapestiensis de Rolando Eötvös nominatae—Sectio Mathematica	Hungary
Annu. Rev. Physiol.	Annual Review of Physiology	U.S.A.
Aplik. Mat.	Aplikace Matematiky	Czechoslovakia
Appl. Sci. Res., Hague	Applied Scientific Research, The Hague	Netherlands
Appl. Statist.	Applied Statistics	Great Britain
Archimede	Archimede	Italy
Arch. Biochem. Biophys.	Archives of Biochemistry and Biophysics	U.S.A.
Arch. Math., Karlsruhe	Archiv der Mathematik	Germany
Ark. Mat.	Arkiv för Matematik	Sweden
Atti (mem.) Accad. Lincei	Atti della Accademia Nazionale dei Lincei	Italy
Atti Riun. Sci., Soc. Ital. Statist.	Atti della Riunione Scientifica, Società Italiana di Statistica, Rome	Italy
Aust. J. Agric. Res.	Australian Journal of Agricultural Research	Australia
Aust. J. Appl. Sci.	Australian Journal of Applied Science	Australia
Aust. J. Biol. Sci.	Australian Journal of Biological Sciences	Australia
Aust. J. Phys.	Australian Journal of Physics	Australia
Aust. J. Sci.	Australian Journal of Science	Australia
Aust. J. Statist.	Australian Journal of Statistics	Australia

Beil. Mber. Öst. Inst. Wirtschaftsf.	Beilagen zu den Monatsberichten des Österreichischen Institutes für Wirtschaftsforschung	Austria
Bell Lab. Record	Bell Laboratories Record	U.S.A.
Bell Syst. Monographs	Bell System Monographs	U.S.A.
Bell Syst. Tech. J.	Bell System Technical Journal	U.S.A.
Biometrics	Biometrics	U.S.A.
Biometrika	Biometrika	Great Britain
Biom. Zeit.	Biometrische Zeitschrift	Germany
Blä. Dtsch. Ges. Versich.-math.	Blätter der Deutschen Gesellschaft für Versicherungsmathematik	Germany
Bol Fac. Filos. Ciênt., S. Paulo	Boletim da Faculdade de Filosofía, Ciências e Letras, São Paulo	Brazil
Bol. Inst. Actuar. Port.	Boletim do Instituto dos Actuarios Portugueses	Portugal
Bol. Soc. Mat. Mexicana	Boletim de la Sociedad Matemática Mexicana	Mexico
Boll. Cent. Ric. Operat.	Bollettino del Centro per la Ricerca Operativa	Italy
Boll. Un. Mat. Ital.	Bollettino dell'Unione Matematica Italiana	Italy
Brit. J. Nutr.	British Journal of Nutrition	Great Britain
Brit. J. Prev. Soc. Med.	British Journal of Preventive and Social Medicine	Great Britain
Brit. J. Sociol.	British Journal of Sociology	Great Britain
Brit. J. Statist. Psychol.	British Journal of Statistical Psychology	Great Britain
Bull. Acad. Polon. Sci. III	Bulletin de l'Académie des Sciences (Classe 3)	Poland
Bull. Amer. Soc. Test. Mat.	Bulletin, American Society for Testing Materials	U.S.A.
Bull. Ass. Actuair. Suisses	Bulletin de l'Association des Actuaires Suisses	Switzerland
Bull. Calcutta Math. Soc.	Bulletin of the Calcutta Mathematical Society	India
Bull. Calcutta Statist. Ass.	Bulletin, Calcutta Statistical Association	India
Bull., C.S.I.R.O., Aust.	Bulletins, Commonwealth Scientific and Industrial Research Organisation	Australia
Bull. Indian Soc. Qual. Contr.	Bulletin, Indian Society of Quality Control	India
Bull. Indian Stand. Inst.	Bulletin, Indian Standards Institution	India
Bull. Inst. Égypte	Bulletin de l'Institut d'Égypte, Cairo	Egypt
Bull. Jap. Statist. Soc.	Bulletin of the Japan Statistical Society	Japan
Bull. Math. Statist.	Bulletin of Mathematical Statistics, Research Association of Statistical Sciences	Japan
Bull. Nat. Inst. Sci. India	Bulletin, National Institute of Sciences of India	India
Bull. Oxf. Univ. Inst. Statist.	Bulletin of the Oxford University Institute of Statistics	Great Britain
Bull. Res. Inst. Kerala	Bulletin, Central Research Institute, Kerala (University)	India
Bull. Sci. Math.	Bulletin des Sciences Mathématique	France
Bull. Soc. Math. Belg.	Bulletin de la Société Mathématique de Belgique, Gembloux	Belgium

Bull. Soc. Math. France	Bulletin de la Société Mathématique de France	France
Bull. Statist. Soc., N.S.W.	Bulletin of the Statistical Society, N.S. Wales	Australia
Canad. J. Econ. Polit. Sci.	Canadian Journal of Economic and Political Science	Canada
Cancer, New York	Cancer (New York)	U.S.A.
Cas. pěst. math.	Casopis pro pěstování mathematiky	Czechoslovakia
Chem. Engng. Progr.	Chemical Engineering Progress	U.S.A.
Clin. Chem.	Clinical Chemistry	U.S.A.
Colect. Estud.	Colectanea de Estudos	Spain
Coll. Mat.	Collectanea Matematica	Spain
Colloq. Math.	Colloquium Mathematicum	Poland
Comment. Math. Helvetia	Commentarii Mathematica Helvetia	Switzerland
Comment. Pontif. Acad. Sci.	Commentationes Pontificiae Academiae Scientiarum	Italy
Compos. Math., Groningen	Compositio Mathematica, Groningen	Netherlands
C.R. Acad. Sci., Paris	Comptes Rendus de l'Académie des Sciences, Paris	France
Cornell Vet.	Cornell Veterinarian	U.S.A.
Current Science	Current Science	India
Curso Mira Fernandes	Curso Mira Fernandes	Portugal
Czech. Math. J.	Czechoslowak Mathematical Journal	Czechoslovakia
Dacca Univ. Statist. Bull.	Dacca University Statistical Bulletin	Pakistan
Defense Sci. J.	Defense Science Journal	India
Dokl. Akad. Nauk. SSSR	Doklady Akademiia Nauk, SSSR.	U.S.S.R.
Dopov. Akad. Nauk. Ukrain, SSR	Dopovedy Akademiia Nauk Ukrainskoj, SSR	U.S.S.R.
Écon. Appl.	Économie Appliquée	France
Econ. Internazionale	Economia Internazionale	Italy
Econ. J.	Economic Journal	Great Britain
Écon. Leban. Arabe	L'Économie Lebanaise & Arabe, Beirut	Lebanon
Econ. Record	Economic Record	Australia
Econometrica	Econometrica	U.S.A.
Economica	Economica	Great Britain
Edinb. Math. Notes	Edinburgh Mathematical Notes	Great Britain
Égypte Contemp.	L'Égypte Contemporaine, Cairo	Egypt
Egyptian Statist. J.	Egyptian Statistical Journal	Egypt
Ekon. Tidskr.	Ekonomisk Tidskrifts	Sweden
Emp. J. Exp. Agric.	Empire Journal of Experimental Agriculture	Great Britain
Endocrinology	Endocrinology	U.S.A.
Engineer, Lond.	Engineer	Great Britain
Engineering, Lond.	Engineering	Great Britain
Ergebn. Naturw. Grenzgeb.	Ergebnisse der exakten Naturwissenschaften	Germany
Esta	Esta	Spain
Estadistica	Estadistica	U.S.A.
Euclides, Groningen	Euclides, Groningen	Netherlands
Eugen. Rev.	Eugenics Review	Great Britain
Evolution	Evolution	U.S.A.
Food Res.	Food Research	U.S.A.
Food Tech.	Food Technology	U.S.A.
Ganita	Ganita, Lucknow-Allahabad (Universities)	India
Gaz. Mat.	Gazeta de Matemática	Portugal
G. Economisti	Giornale degli Economisti e Annali di Economia	Italy
G. Ist. Ital. Attuari	Giornale dell'Istituto Italiano degli Attuari	Italy
Genetics	Genetics	U.S.A.
Genus	Genus	Italy
Growth	Growth	U.S.A.
Harvard Busin. Rev.	Harvard Business Review	U.S.A.
Helvetica Phys. Acta	Helvetica Physica Acta	Switzerland

Heredity	Heredity	Great Britain
Hum. Biol.	Human Biology	U.S.A.
IFO-Studien	IFO-Studien	Germany
Inc. Statist.	Incorporated Statistician	Great Britain
Indag. Math.	Indagationes Mathematicae, Amsterdam	Netherlands
Indian Coun. Agric. Res. Bull.	Indian Council of Agricultural Research : Bulletin	India
Indian Coun. Agric. Res. Statist. Newsletter	Indian Council of Agricultural Research : Statistical Newsletter	India
Indian Econ. J.	Indian Economic Journal	India
Indian Econ. Rev.	Indian Economic Review	India
Indian J. Agric. Econ.	Indian Journal of Agricultural Economics	India
Indian J. Phys.	Indian Journal of Physics	India
Indian J. Power & River Devel.	Indian Journal of Power and River Development	India
Indian J. Soc. Work	Indian Journal of Social Work	India
Indian Math. Soc. J.	Indian Mathematical Society Journal	India
Industr. Engng. Chem.	Industrial and Engineering Chemistry	U.S.A.
Industr. Qual. Contr.	Industrial Quality Control	U.S.A.
Industria, Milano	L'Industria	Italy
Iowa Acad. Sci. J.	Iowa Academy of Science Journal	U.S.A.
Iowa Exp. Sta. Publ.	Iowa State College Experiment Station Publications	U.S.A.
Izv. Akad. Nauk, SSSR	Izvestiya Akademiia Nauk, SSSR	U.S.S.R.
Izv. Akad. Nauk, Uzbek., SSR.	Izvestiya Akademiia Nauk, Uzbekskoj, SSR.	U.S.S.R.
Jb. Nat.-Ökon. Statist.	Jahrbücher für Nationalökonomie und Statistik	Germany
Jber. Dtsch. Mathver.	Jahresbericht der Deutschen Mathematikervereinigung	Germany
J. Agric. Food Chem.	Journal of Agricultural and Food Chemistry	U.S.A.
J. Agric. Sci.	Journal of Agricultural Science	Great Britain
J. Amer. Pharm. Ass.	Journal of the American Pharmaceutical Association (Scientific Edition)	U.S.A.
J. Amer. Statist. Ass.	Journal, American Statistical Association	U.S.A.
J. Amer. Vet. Med. Ass.	Journal of American Veterinary Medicine Association	U.S.A.
J. Animal Sci.	Journal of Animal Sciences	U.S.A.
J. Appl. Physiol.	Journal of Applied Physiology	U.S.A.
J. Ass. Comp. Mach.	Journal of the Association of Computing Machinery	U.S.A.
J. Aust. Inst. Agric. Sci.	Journal of the Australian Institute of Agricultural Science	Australia
J. Aust. Math. Soc.	Journal of the Australian Mathematical Society	Australia
J. Biol. Chem.	Journal of Biological Chemistry	U.S.A.
J. Chronic Diseases	Journal of Chronic Diseases	U.S.A.
J. Clin. Invest.	Journal of Clinical Investigation	U.S.A.
J. Dairy Sci.	Journal of Dairy Science	U.S.A.
J. Ecology	Journal of Ecology	Great Britain
J. Econ. Ent.	Journal of Economic Entomology	U.S.A.
J. Exp. Educ.	Journal of Experimental Education	U.S.A.
J. Exp. Med.	Journal of Experimental Medicine	U.S.A.
J. Forestry	Journal of Forestry	U.S.A.
J. Gen. Physiol.	Journal of General Physiology	U.S.A.
J. Genet.	Journal of Genetics	India
J. Heredity	Journal of Heredity	U.S.A.
J. Hyg., Camb.	Journal of Hygiene, Cambridge	Great Britain
J. Industr. Econ.	Journal of Industrial Economics	Great Britain
J. Industr. Engng.	Journal of Industrial Engineering	U.S.A.
J. Indian Soc. Agric. Statist.	Journal of the Indian Society of Agricultural Statistics	India
J. Inst. Actuar.	Journal of the Institute of Actuaries	Great Britain
J. Inst. Actuar. Stud. Soc.	Journal of the Institute of Actuaries Students' Society	Great Britain
J. Madras Univ., B	Journal of Madras University, Section B.	Great Britain
J. Mammalogy	Journal of Mammalogy	U.S.A.
J. Marine Res.	Journal of Marine Research	U.S.A.
J. Math. Pures Appl.	Journal de Mathématiques Pures et Appliquées	France
J. Math. Soc. Japan	Journal of the Mathematical Society of Japan	Japan
J. Nutrition	Journal of Nutrition	U.S.A.
J. Operat. Res.	Journal of the Operations Research Society of America	U.S.A.

J. Pharmacol.	Journal of Pharmacology and Experimental Therapeutics	U.S.A.
J. Plant Physiol.	Journal of Plant Physiology	U.S.A.
J. Polit. Econ.	Journal of Political Economy	U.S.A.
J. reine angew. Math.	Journal für reine und angewandte Mathematik	Germany
J. Res., Nat. Bur. Stand.	Journal of Research, National Bureau of Standards, Washington	U.S.A.
J. Roy. Soc., N.S.W.	Journal of the Royal Society, N.S. Wales	Australia
J. R. Statist. Soc. A & B	Journal of the Royal Statistical Society (Series A & B)	Great Britain
J. Sci. Engng.	Journal of Science and Engineering	India
J. Sci. Industr. Res.	Journal of Scientific and Industrial Research	U.S.A.
J. Sci. Industr. Res., India	Journal of Scientific and Industrial Research	India
J. Soc. Statist. Paris	Journal de la Société Statistique de Paris	France
J. Statist. Social Inquiry Soc.	Journal of the Statistical and Social Inquiry Society of Ireland	Ireland
J. Univ. Baroda	Journal, Maharaja Sayajiroa, Baroda University	India
Kōdai, Tokyo Inst. Tech.	Kōdai Mathematical Seminar reports, Tokyo Institute of Technology	Japan
Magy. Tud. Akad. III *Oszt. Közl.*	Magyar Tudományos Akadémia Matematikai és Fizikai Osztályának Közleményei	Hungary
Management Sci.	Management Science	U.S.A.
Manager (India)	Manager	India
Manchester School	Manchester School	Great Britain
Matemática	Matemática, Notas del Instituto de Matemática Facultad de Ingeniería de Rosario	Argentine
Mat. Lapok	Matematikai Lapok	Hungary
Math. Ann.	Mathematische Annalen	Germany
Math. Gaz.	Mathematical Gazette	Great Britain
Math. Nachr.	Mathematische Nachrichten	Germany
Math. Stud.	Mathematics Students	India
Math. Tab., Wash.	Mathematical Tables and other aids to Computation	U.S.A.
Math., Tech., Wirtschaft	Mathematik, Technik, Wirtschaft (Zeitschrift für moderne Rechentechnik und Automation)	Austria
Math. Zeit.	Mathematische Zeitschrift	Germany
Meded. Landbhoogesch., Wageningen	Mededeelingen van de Landbouwhoogeschool, Wageningen	Netherlands
Mem. Coll. Sci., Kyoto	Memoirs of the College of Science, Kyoto (University)	Japan
Mem. Fac. Sci. A, *Kyusyu*	Memoirs of the Faculty of Science (Series A), Kyusyu (University)	Japan
Mém. Soc. Sci. Liége	Mémoires de la Société Royale des Sciences de Liége	Belgium
Metabolism	Metabolism	U.S.A.
Met. Mag., Lond.	Meteorological Magazine	Great Britain
Metrika	Metrika	Germany
Metroeconomica	Metroeconomica	Italy
Metron	Metron	Italy
Mitt. Ver. Schweiz. Versich.-Math.	Mitteilungen der Vereinigung Schweizerisher Versicherungs-Mathematiker	Switzerland
Monatsh. Math.	Monatshefte für Mathematik	Austria
Monthly Not. R. Astr. Soc.	Monthly Notices of the Royal Astronomical Society	Great Britain
Nat. Bur. Stand., Statist. Rep. & Publ.	National Bureau of Standards, Washington : Statistical Reports and Publications	U.S.A.
Nature	Nature (London)	Great Britain
Ned. Verzek. Actuar. Bij.	Nederlands Verzekeringsarchief, Actuarieel Bijvoegsel, The Hague	Netherlands
Nieuw Arch. Wisk.	Nieuw Archief voor Wiskunde, Groningen	Netherlands
Nieuw Tijdschr. Wisk.	Nieuw Tijdschrift voor Wiskunde, Groningen	Netherlands

Operat. Res. Quart.	Operational Research Quarterly	Great Britain
Osaka Math. J.	Osaka Mathematical Journal, Osaka (University)	Japan
Öst. Ingen.-arch.	Österreichisches Ingenieurarchiv	Austria
Oxf. Econ. Papers	Oxford Economic Papers	Great Britain
Pakistan Econ. J.	Pakistan Economic Journal	Pakistan
Phil. Mag.	Philosophical Magazine	Great Britain
Phil. Phenom. Res.	Philosophical and Phenomenological Research	Great Britain
Phil. Trans.	Philosophical Transactions of the Royal Society	Great Britain
Population Stud.	Population Studies	Great Britain
Portug. Math.	Portugaliae Mathematica	Portugal
Poultry Sci.	Poultry Science	U.S.A.
Proc. Amer. Soc. Hort. Sci.	Proceedings, American Society for Horticultural Science	U.S.A.
Proc. Amer. Soc. Qual. Contr.	Proceedings of the American Society for Quality Control	U.S.A.
Proc. Amer. Soc. Test. Mat.	Proceedings of the American Society for Testing Materials	U.S.A.
Proc. Camb. Phil. Soc.	Proceedings of the Cambridge Philosophical Society	Great Britain
Proc. Indian Acad. Sci.	Proceedings of the Indian Academy of Sciences	India
Proc. Indian Ass. Cult. Sci.	Proceedings of the Indian Association for Cultivation of Science	India
Proc. Indian Sci. Congr.	Proceedings of the Indian Science Congress	India
Proc. Inst. Statist. Math.	Proceedings of the Institute of Statistical Mathematics	Japan
Proc. Japan Acad.	Proceedings of the Japan Academy	Japan
Proc. Kon. Ned. Akad., Wetensch. A	Proceedings, Koninklijke Nederlandsche Akademie van Wetenschappen (A)	Netherlands
Proc. Lahore Phil. Soc.	Proceedings of the Lahore Philosophical Society	India
Proc. Lond. Math. Soc.	Proceedings of the London Mathematical Society	Great Britain
Proc. Math. Phys. Soc. Egypt	Proceedings of the Mathematical and Physical Society of Egypt	Egypt
Proc. Nat. Acad. Sci. India	Proceedings of the National Academy of Sciences	India
Proc. Nat. Inst. Sci. India (A & B)	Proceedings of the National Institute of Sciences of India (Section A & B)	India
Proc. Pakistan Statist. Ass.	Proceedings of the Pakistan Statistical Association	Pakistan
Proc. Qual. Contr. Ass., Bangalore	Proceedings of the Quality Control Association, Bangalore	India
Proc. R. Irish Acad.	Proceedings of the Royal Irish Academy	Ireland
Proc. Roy. Soc. (A & B)	Proceedings of the Royal Society (Series A & B)	Great Britain
Proc. Roy. Soc. N.S.W.	Proceedings of the Royal Society of N.S. Wales	Australia
Proc. Roy. Soc. N.Z.	Proceedings of the Royal Society of New Zealand	New Zealand
Proc. Soc. Exp. Biol., N.Y.	Proceedings of the Society of Experimental Biology and Medicine	U.S.A.
Proc. Soil Sci. Soc. Amer.	Proceedings, Soil Science Society of America	U.S.A.
Przegl. Statyst.	Przeglad Statystyczny	Poland
Psychometrika	Psychometrika	U.S.A.
Public Health Rep.	Public Health Reports	U.S.A.
Public Opinion Quart.	Public Opinion Quarterly	U.S.A.
Publ. Inst. Mat. Estadíst. Fac. Ingen. Agric., Montevideo	Publicaciones del Instituto de Matemáticas y Estadística de la Facultad de Ingenieria y Agrimensura, Montevideo	Uruguay
Publ. Inst. Statist. Paris	Publications de l'Institut de Statistique de l'Université de Paris	France
Publ. Math., Debrecen	Publicationes Mathematicae	Hungary
Publ. Math. Inst. Hung. Acad. Sci.	Publications of the Mathematical Institute of the Hungarian Academy of Sciences	Hungary
Quart. J. Econ.	Quarterly Journal of Economics	U.S.A.
Quart. J. Exp. Physiol.	Quarterly Journal of Experimental Physiology	U.S.A.
Quart. J. Math.	Quarterly Journal of mathematics, Oxford	Great Britain
Quart. J. Med.	Quarterly Journal of Medicine	Great Britain
Quart. J. R. Met. Soc.	Quarterly Journal of the Royal Meteorological Society	Great Britain

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R.C. Semin. Sci. Cagliari	Rendiconti del Seminario della Facoltà di Scienze, Cagliari (University)	Italy
Rep. Statist. Appl. Res. (JUSE)	Reports of Statistical Application Research, Union of Japanese Scientists & Engineers	Japan
Research, Beirut	Research, A.U.B.—Beirut	Lebanon
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Rev. Brasil. Estatist.	Revista Brasileira de Estatística	Brazil
Rev. Cienc. Tucumán	Revista de Ciencias de la Universidad Nacional del Tucumán	Argentine
Rev. Écon.	Revue Économique	France
Rev. Econ. Statist.	Review of Economics and Statistics	U.S.A.
Rev. Econ. Stud.	Review of Economic Studies	Great Britain
Rev. Economia, Lisbon	Revista de Economia	Portugal
Rev. Estocást. Soc. Argent. Estadíst.	Revista Estocástica de la Sociedad Argentina de Estadística	Argentine
Rev. Fac. Cienc. Econ. Admin. Montevideo	Revista de la Facultad de Ciencias Economicas y de Administracion, Montevideo	Uruguay
Rev. Int. Statist. Inst.	Review, International Statistical Institute	Netherlands
Rev. Mat. Hisp.-Amer.	Revista Matematica Hispano-Americana	Spain
Rev. Rech. Operat.	Revue de Recherche Operationelle	France
Rev. Sci., Paris	Revue Scientifique	France
Rev. Soc. Cubana Cienc. Fìs. Mat.	Revista de la Sociedad Cubana de Ciencas, Fìsicas y Matemàticas	Cuba
Rev. Statist. Appl.	Revue de Statistique Appliquée	France
Ric. Sci.	Ricerca Scientifica	Italy
Riv. Int. Sci. Soc.	Rivista Internazionale di Science Sociali	Italy
Riv. Ital. Econ. Demogr. Statist.	Rivista Italiana di Economia Demografia e Statistica	Italy
Riv. Polit. Econ.	Rivista di Politica Economia	Italy
Sankhyā	Sankhyā; Indian Journal of Statistics	India
Schmollers Jb.	Schmollers Jahrbuch	Germany
Schr. Ver. Sozialpolit.	Schriften des Vereins für Sozialpolitik	Germany
Schweiz. Zeit. Volkwirtsch. Statist.	Schweizerische Zeitschrift für Volkwirtschaft und Statistik	Switzerland
Science	Science	U.S.A.
Science & Culture	Science and Culture	India
Scientia	Scientia	Italy
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Simon Stevin	Simon Stevin, Groningen	Netherlands
Skand. Aktuartidskr.	Skandinavisk Aktuarietidskrift	Sweden
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South Africa J. Econ.	South African Journal of Economics	South Africa
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Studia Math.	Studia Mathematica	Poland
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Tech. News, Nat. Bur. Stand.	Technical News Letter, National Bureau of Standards, Washington	U.S.A.
Tech. Publ., Amer. Soc. Test. Mat.	Technical Publications, American Society for Testing Materials	U.S.A.
Tech. Rep., C.S.I.R.O., Aust.	Technical Reports, Commonwealth Scientific & Industrial Research Organisation	Australia

Technometrics	Technometrics	U.S.A.
Teor. Veroyat. Primen.	Teoriia Veroyatnostey i ee Primenenia	U.S.S.R.
Textile Res.	Textile Research	U.S.A.
Tobacco Sci.	Tobacco Science	U.S.A.
Trab. Estadíst.	Trabajos de Estadística	Spain
Trans. Edin. Math. Soc.	Transactions of the Edinburgh Mathematical Society	Great Britain
Trans. Fac. Actuar., Edinb.	Transactions of the Faculty of Actuaries	Great Britain
Trans. Manchester Statist. Soc.	Transactions of the Manchester Statistical Society	Great Britain
Trans. Nat. Inst. Sci. India	Transactions of the National Institute of Sciences of India	India
Trans. N.Y. Acad. Sci.	Transactions of the New York Academy of Science	U.S.A.
Trans. Roy. Soc. N.Z.	Transactions of the Royal Society, N. Zealand	New Zealand
Trans. Soc. Actuar., Chicago	Transactions of the Society of Actuaries	U.S.A.
Ukrain. Mat. Z.	Ukrainskij Matematiceskij Zurnal	U.S.S.R.
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U.S. Dept. Agric., Publ.	U.S. Department of Agriculture, Publications	U.S.A.
Verh. Kon. Belg. Akad., II	Verhandelingen. Koninklijke Belgische Akademie (Serie 2)	Belgium
Verh. Vlaamse Akad. Belg.	Verhandelingen. Koninklijke Vlaamsche Akademie	Belgium
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Weltwirts. Arch.	Weltwirtschaftliches Archiv	Germany
Wirtsch. und Statist.	Wirtschaft und Statistik	Germany
Yorkshire Bull. Econ. Soc. Res.	Yorkshire Bulletin of Economic and Social Research	Great Britain
Zastosowania Mat.	Zastosowania Matematyki	Poland
Zeit. angew. Math. Mech.	Zeitschrift für angewandte Mathematik und Mechanik	Germany
Zeit. ges. Staatswiss.	Zeitschrift für die gesamte Staatswissenschaft	Germany
Zeit. Handelswiss. Forsch.	Zeitschrift für Handelswissenschaftliche Forschung	Germany
Zeit. Nat.-Ökon.	Zeitschrift für Nationalökonomie	Austria

The short titles given above are based upon the principles stated in the League of Nations report *International Code of Abbreviations for Titles of Periodicals* (1930) and its *Supplement* (1932) as also used in the *World List of Scientific Periodicals*

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CONTENTS

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VOL. 1: No. 4 (533-751)

Volume 1. CORRECTIONS

35	**3.1**	Darwin	line 9—delete " varying "; lines 20, 23—read " transform " for " transformation ".
37	**3.5**	Hooper	line 15—substitute $p = \xi\xi'$
77	**6.1**	Barton & Casley	line 19—read " . . . moments of the estimate . . . "
108	**8.1**	Stevens	line 8—for N read n.
142	**10.1**	Smith	reference at end should be to " Wold and Domb's work . . . ". The author's address should read " University of North Carolina, Chapel Hill ".
296	**10.5**	Blyth	page numbers should read 71-79.
519	**10.4**	Winsten	serial number in Editorial Note should read 520.
520	**10.4**	Winsten	serial number in Editorial Note should read 519.
Author Index No. 2			Mendenhall, W. should read " 54, 155 ".
Author Index No. 3			Add : Arens, B. E. **6.1**; David, H. A. **6.1**; Guttman, I. 2.8, **4.5**; Ishii, G. 5.6
			Amend : Wilkinson, R. A. to Wijsman, R. A.
			Add to : Graybill, F. A. the entry " **4.1** ".
			Delete : **6.1** from David, Florence N.; 2.8, **4.5** from Guttman, L.; 5.6 from Isii, K.
			The final entry should read " Zubrzycki, S."

GINI, C. (University of Rome) **0.0** (11.0)

Mathematics in statistics—*In English*

Metron (1959) **19**, 3-9 (2 references)

In this short paper the author gives an account of the value of mathematics for statistics. This article deals first of all with the subjectivity of the concept of mathematics in statistics. For the student of letters or of law the square root that appears in the expression for the standard deviation is already higher mathematics, while the modern statistician, when considering mathematics in statistics, is thinking of algebra and calculus.

The author insists on considering mathematics as a means and not as an end for statistics: therefore while the professional mathematician seldom limits his developments to that which is strictly necessary, his methods are comprehensible only to a limited number of specialists. The author comments that it would be good to substitute the slogan " Statistics *with* Mathematics " for " Statistics *like* Mathematics " or (further) with " Statistics with the simplest mathematics possible ".

In section four of this note the method of " models " is illustrated in general: their character of compromise between the deductive method of mathematics and the inductive one of statistics is shown. The delicate problem of direct and inverse probability is the particular object of the last section of this paper. In this connection reference should be made to " Direct problems and inverse problems of statistical research " being chapter 15 of *Statistical Methods with special reference to Agriculture* [Int. Center for training in Agricultural Economics and Statistics (1956) Rome].

The author concludes his note with some considerations of statistical relations and stochastic relations. On this particular argument he proposed some years ago the use of special symbols which would allow of distinguishing the variable from the function in statistical relations, individual values from average values, cograduated values from corresponding values and which would also indicate the frequency with which correspondence between the values of the variables and of the function occurs. [See " On some symbols that may be usefully employed in Statistics ", *Bull. Int., Statist. Inst.* (1951) **33**, 249-282 and *Metron* (1953) **17**].

(C. Benedetti)

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GREENBERG, B. G. & SARHAN, A. E. (University of North Carolina, Chapel Hill) **0.6** (9.3)

Matrix inversion, its interest and application in analysis of data—*In English*

J. Amer. Statist. Ass. (1959) **54**, 755-766 (10 references)

Matrix inversion is used in the least-squares analysis of data to estimate parameters and their variances and covariances. When the data come from the analysis of variance, analysis of covariance, order statistics, or the fitting of response-surfaces, the matrix to be inverted usually falls into a structured pattern that simplifies its inversion.

One class of patterned matrices is characterised by non-singular symmetrical arrangements in which linear combinations of the first $(r-1)$ rows provide the right-hand portion, starting with the elements on the principal diagonal, of the rth and remaining rows. That is:

$$\theta_{1i}v_{1j}+\theta_{2i}v_{2j}+\ldots+\theta_{r-1,i}v_{r-1,j} = v_{ij}$$

for $r \leqq i \leqq j$, with $v_{ij} = v_{ji}$ for all $i \neq j$. The inverses of matrices of this class contain a non-null principal diagonal, and immediately adjacent to the principal diagonal, $(r-1)$ non-null superdiagonals and $(r-1)$ non-null subdiagonals. All other elements are zero. These inverses are called diagonal matrices of type r. That is, a matrix is diagonal of type r if $\alpha_{ij} = 0$ for $|i-j| \geqq r$ and $\alpha_{ij} = \alpha_{ji}$ for all $i \neq j$. When $r = 2$, the inverse is easily written in terms of θ_{1i} and v_{ij}. A general procedure for obtaining the inverse when $r = 3$ is given.

The results for $r = 2$ are illustrated by a problem in order statistics using the two-parameter exponential distribution.

Patterned matrices also are amenable to partitioning and this is another convenient device to find the exact inverse quickly. In fitting a response-surface to data, for example, a complicated-looking matrix can be abbreviated and simplified by selective partitioning. When this has been done, the exact inverse can be found by equating the product of the matrix and its inverse to the elements of the identity matrix. This procedure is illustrated with some data from a problem in the fitting of a response-surface.

When the matrix has no special pattern, as in the usual regression problem, the recommended procedure for matrix inversion is the modified square-root method.

(B. G. Greenberg)

MADANSKY, A. (RAND Corporation, St. Monica, Cal.) **0.5** (6.6)

The fitting of a straight line when both variables are subject to error—*In English*

J. Amer. Statist. Ass. (1959) **54,** 812

The author notifies a brief list of five corrections in his paper already abstracted in this journal, No. 162, **0.5**. The page numbers affected are 183, 187 and 191.

(W. R. Buckland)

VOLPATO, M. (Faculty of Economics, University of Venice) **0.8** (–.–)

On *a priori* bounding of the extrema of the object function in problems of linear programming—*In Italian*

Ricerche Economiche (1959) **13**, 363-377 (3 references)

The purpose of the author is to give an inferior and superior boundary to the values that the object function

$$\overset{n}{\underset{1}{\Sigma_s}}\, c_s x_s$$

can assume in the admissible region A, which comprises all points $X \equiv (x_1, \dots, x_n)$ of the real Euclidean space S_n, whose coordinates satisfy the (independent) equalities

$$\overset{n}{\underset{1}{\Sigma_s}}\, a_{\tau s} x_s = b_\tau, \quad (\tau = 1, \dots, m),$$

and the conditions of non-negativity.

For this purpose, the author assumes that in the real Euclidean space at least a point $Y \equiv (y_1, \dots, y_m)$ exists whose coordinates satisfy, indifferently, the inequalities

$$\overset{m}{\underset{1}{\Sigma_\tau}}\, a_{\tau s} y_s \geqq 0, \quad (s = 1, \dots, n), \quad \overset{m}{\underset{1}{\Sigma_\tau}}\, b_\tau y_\tau \neq 0,$$

or the inequalities

$$\overset{m}{\underset{1}{\Sigma_r}}\, a_{rs} y_s \leqq 0, \quad (s = 1, \dots, n), \quad \overset{m}{\underset{1}{\Sigma_r}}\, b_r y_r \neq 0,$$

This assumption is not too restrictive for practical applications.

He then shows that, for any point $X \equiv (x_1, \dots, x_n)$ of the admissible region A and for any point $Y \equiv (y_1, \dots, y_m)$ satisfying either of the above inequalities, the following inequality is valid

$$\inf_s \left(\frac{c_s \overset{n}{\underset{1}{\Sigma_\tau}}\, b_\tau y_\tau}{\overset{m}{\underset{1}{\Sigma_r}}\, a_{\tau s} y_\tau} \right) \leq \overset{n}{\underset{1}{\Sigma_s}}\, c_s x_s \leq \sup_s \left(\frac{c_s \overset{m}{\underset{1}{\Sigma_\tau}}\, b_\tau y_r}{\overset{m}{\underset{1}{\Sigma_r}}\, a_{rs} y_r} \right)$$

with a convenient interpretation of the ratios when these should be indeterminate.

The knowledge of a pair of bounding values as given by last inequalities mentioned above can sometimes be enough to determine the choice of a decision. It is appropriate to observe that if a certain problem of linear programming satisfying the stated hypothesis is the analytic representation of an economic, physical or technical problem for which one wants to determine the optimum programming, then the value of the object function corresponding to the given programme must necessarily be included between the extrema of the third set of inequalities given above. Should this not happen, then one is certain that the analytic representation does not adequately reproduce the problem and its conditions.

(G. Panizzon)

ZOROA, P. (Instituto de Investigaciones Estadísticas, Madrid) **0.7** (0.6)

Saddle matrix in rectangular games—*In Spanish*

Trab. Estadíst. (1959) **10**, 3-12

The saddle matrix in a finite rectangular game is defined as a generalisation of the saddle point; that is, as a submatrix of the matrix of the game such that linear convex combinations of its rows dominate the remaining rows without considering the columns not belonging to the considered submatrix: a symmetric condition on the columns is also given. The search for the solution for a game which has a saddle matrix is then reduced to finding the solution for a finite game which has as its matrix the desired saddle matrix. An optimal strategy for the original game is given by completing with zeros the vector which gives an optimal strategy for the saddle matrix.

The problem of obtaining an optimal strategy for the game whose matrix is the saddle matrix, provided that a solution for the original game is known, is also considered.

(S. Rios)

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ZOUTENDIJK, G. (Koninklijke/Shell-Lab., Amsterdam) **0.8** (11.5)

Maximising a function in a convex region—*In English*

J. R. Statist. Soc. B (1959) **21**, 338-355 (19 references)

The author is primarily concerned with maximising a function subject to linear constraints by a method that he calls the " method of feasible directions ". Starting from a trial feasible solution, he presents efficient methods (using the simplex technique) for finding the best feasible solution inside an arbitrarily small hypersphere (Procedure S1) or hypercube (Procedure S2) centred on the trial solution. This defines a direction in which to move, and the trial solution is moved as far as is both feasible and profitable in this direction. The process is then repeated. It converges to a local maximum of the objective function, which must be a global maximum if this function is concave.

The problem is formulated and the method of solution outlined in the second section. In the third section it is shown to converge, in a finite number of steps for linear problems or an infinite number of steps for non-linear problems, if a specified anti-zigzagging precaution is used.

The fourth section considers in more detail the method of finding the directions in which to move the trial solution. A special dual simplex algorithm is presented for solving the simple quadratic programming problem that arises with Procedure S1. In the next section reasons are suggested why these methods may be preferable to the simplex method for linear problems and in the sixth section. A modification of the method is presented for concave quadratic objective functions that uses conjugate directions to give a finite algorithm.

The author next modifies the method (section 7) to deal with nonlinear constraints. If a trial solution lies on a non-linear constraint, the method required the chosen feasible direction to move away from this constraint. Otherwise one might only be able to move a zero distance without violating the constraint.

Finally, comparisons are made with other methods, particularly for the problem with linear constraints and a concave quadratic objective function.

(E. M. L. Beale)

BENEDETTI, C. (Faculty of Statistics, University of Rome) **1.8** (3.7)

On some criteria of statistical decision based on the solution of problems of minimum distance—*In Italian*

Metron (1959) **19**, 22-37 (2 references, 4 tables)

In this article the author solves the following problem: let Ω_n be the closed convex set of all points in n-dimensional Euclidean space E_n so that the coordinates x_i of every point $x = (x_1, x_2, \ldots x_n) \in \Omega_n$ are non-decreasing $(x_1 \leqq x_2 \leqq \ldots \leqq x_n)$ and let there be in E_n a point $A \notin \Omega_n$, then the point $\bar{x} \in \Omega_n$ is determined so that: $d(a, \bar{x}) = d(a, \Omega_n) = \min\limits_{x \in \Omega_n} d(a, x)$ where $d(x, y)$ is the Euclidean metric $[\Sigma(x-y)^2]^{\frac{1}{2}}$.

The method that leads to the point $\bar{x}$ of the boundary of Ω_n is iterative, but it converges rapidly. The theorem proved in this paper is as follows:

Let $a = a_1, a_2, \ldots a_n$ be a succession of reals constituting the coordinates of the point $a \notin \Omega_n$, if the k-perequated $A^{(k)}$ is the first among the k successions $a^{(1)}, a^{(2)}, \ldots a^{(k)}$ which does not contain s-sequences, then we have:

$$d(a, \Omega_n) = \min_{x \in \Omega_n} d(a, x) = d(a, \bar{x}) = d(a, a^{(k)}) = \sqrt{\sum_{i=1}^{n} \{a_i - a_i^{(k)}\}^2}$$

and the solution of the problem depends on the following three definitions:

(i) Let $a = a_1, a_2, \ldots a_n$ be a succession of reals; we name s-sequence every set constituted by a maximum number $\mathscr{S}$ of successive terms in a, all being non-increasing; the case where the $\mathscr{S}$ terms are all equal is excluded; it is $2 \leqq \mathscr{S} \leqq n$.

(ii) Given the succession a of the first definition, we name 1-perequation the operation S by which we substitute every term of each $\mathscr{S}$-sequence of a by the arithmetic average of the $\mathscr{S}$ component terms, and 1-perequated the succession $a^{(1)}$ of n terms that derives from this transformation. The transformation S leaves unaltered the terms that do not belong to $\mathscr{S}$-sequences.

(iii) The succession $a^{(2)}$ obtained by the application of S to $a^{(1)}$ is named 2-perequated; and in general, k-perequated the succession $a^{(k)}$ obtained by the application of S to $a^{(k-1)}$.

In the last section of this paper some examples are given to illustrate the problem and its solution.

(C. Benedetti)

BLOEMENA, A. R. (Mathematical Centre, Amsterdam) **1.3** (2.9)

On probability distributions arising from points on a graph—*In English*

Statist. Dept., Math. Centre, Report S266A (1960)

A set S of n points, numbered from 1, ..., n, is given and a symmetrical n by n matrix $\mathbf{M} = (m_{ij})$, where $m_{ii} = 0$. If m_{ij} are non-negative integers, the combination $(S, \mathbf{M})$ can be interpreted as a graph without loops, having n points and m_{ij} joins between point i and j. Define $m_i = \sum\limits_j m_{ij}$. A further assumption is $m_i \geqq 1$ for all j.

From the n points two samples are taken. Two cases have been considered. (i) Non-free sampling: from S, r_1 and r_2 points are chosen at random without replacement $(r_1 + r_2 \leqq n)$. The r_1 points are coloured black (B), the r_2 ones white (W). (ii) Free-sampling: n independent trials are performed, each trial resulting in the events B and W with probability p_1 and p_2 respectively $(p_1 + p_2 \leqq 1)$. Point i is allotted the colour indicated by the outcome of the ith trial.

In the report results are given on the stochastic properties of the random variable x_B being the number of BB joins, and of the random variable y being the number of BW joins. The first two moments of these random variables are given; higher moments have been calculated, and asymptotic properties deduced from them. As a specimen we quote one theorem: If r_1 and n tend to infinity, such that $r/n \to \delta$, $0 < \delta < 1$, and if $m_i < c$ for all i and n then $\{x_B - \mathscr{E} x_B\} \sigma(x_B)^{-1}$ tends to the standard normal distribution.

For the case that, for example, r_1 and n tend to infinity and $r/n \to 0$, results are given which indicate convergence towards a compound-Poisson distribution.

All this generalises results by Moran [*J. R. Statist. Soc.* B (1948) **10**, 243-251 and *Proc. Camb. Phil. Soc.* (1947) **43**, 321-328] and of Krishna Iyer [*Biometrika* (1949) **36**, 135-141 and *Ann. Math. Statist.* (1950) **21**, 198-217].

(A. R. Bloemena)

CASTOLDI, L. (University of Cagliari, Sardinia) **1.3** (1.2)

A variant to a probabilistic problem by Banach—*In Italian*

Boll. Cent. Ric. Operat. (1959) **3**, 24-28 (2 references, 1 table)

The following match-box problem in probability is attributed to Banach: " A certain mathematician always carries two match-boxes; every time he wants a match, he selects a box at random. Inevitably a moment occurs when for the first time he finds a box empty. At that moment the other box may contain $r = 0, 1, 2, \ldots$ matches, and we wish to find the corresponding probabilities u_r. Assume that initially each box contained N matches ".

The author considers a variant of this problem and wants to determine the probability that, when making an extraction that empties a box (not when making an extraction that finds a box for the first time empty, as in the above problem), the other box contains $r(r = 1, 2, \ldots, N)$ matches. For this problem he finds the probability distribution $\Pr(N, r)$, the first two moments from the origin and the variance (also with approximate formulae for large N).

An example is given for $\Pr(50, r)$, for which a table of probabilities and of cumulated probabilities, as a function of $r[r = 1(1)\ 30]$, is presented.

(C. Grossi)

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EEDEN, CONSTANCE van & BLOEMENA, A. R. (Mathematical Centre, Amsterdam) **1.3** (2.9

On probability distributions arising from points on a lattice—*In English*

Statist. Dept., Math. Centre, Report S257 (1960)

In this paper the following problem is considered. Given a rectangular lattice of k rows and m columns ($N = km$, $k \leqq m$): from these N points n are selected at random and without replacement. These n points are called black points, the other N-n points are called white points.

The problem considered concerns the joins between black points, where two points are said to be joined if they are adjacent in a horizontal, vertical or diagonal direction.

The points in the rectangle not being equivalent, the lattice is (for $k \geqq 3$) supposed to be wrapped around a torus and the ends joined. Then each point on the torus has joins with 8 other points. For $k = 1$ with $m > 2$ and $k = 2$ with $m > 2$ the lattice is supposed to be wrapped around a cylinder and the ends joined. Then each point on the cylinder has joins with 2 other points for $k = 1$ and with 5 other points for $k = 2$.

The random variable x to be considered is the number of black-black joins. For $k = 1$ the paper contains a description of the well-known solution of this problem. In this case x possesses a hypergeometric distribution, leading asymptotically (for $N \to \infty$) to a normal or a Poisson distribution. For $k \geqq 2$ the following results are obtained:

(i) the exact distribution of x for $k = 2$ with $n = 2, 3, 4$, for $k = 3$ with $n = 2, 3$, and finally $k = m = n = 4$,

(ii) the mean, variance and third moment of x as a function of n and N,

(iii) the probabilities that x takes the values 0 and 1 as a function of n and N for $k = 2$ and $k = 3$,

(iv) the asymptotic distribution of x for $N \to \infty$ with $\frac{n}{N} \to 0$. This asymptotic distribution is a Poisson-distribution. If $\frac{n}{N} \to 1$ for N then the number of white-white joins possesses asymptotically a Poisson-distribution.

If $0 < \lim\limits_{N \to \infty} \frac{n}{N} < 1$ for $N \to \infty$ then the distribution of x is asymptotically normal. The paper does not contain a proof of this statement: it will be proved for a more general lattice in a subsequent paper by Bloemena (see abstract No. 540 **1.3**)

The paper also contains a brief description of results obtained by Moran [*Proc. Camb. Phil. Soc.* (1947) **43**, 321-328 and *J. R. Statist. Soc.* B (1948) **10**, 243-251] and by Krishna Iyer [*Biometrika* (1949) **36**, 135-141 and *Ann. Math. Statist.* (1950) **21**, 198-217] for closely related problems.

(Constance van Eeden)

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ERDÖS, P. & RÉNYI, A. (Eötvös University, Budapest) **1.5** (0.0)

On Cantor's series with convergent $\Sigma\, 1/q_n$—*In English*

Ann. Univ. Sci. Budapest., Sect. Math. (1959) **2**, 93-109 (8 references)

The following generalisation of the Borel-Cantelli lemma is proved: Let $\{A_n\}$ be a sequence of events such that

$$\sum_{n=1}^{\infty} P(A_n) = +\infty$$

and

$$\varliminf_{n\to\infty} \sum_{k=1}^{n}\sum_{l=1}^{n} P(A_k A_l)/[\sum_{k=1}^{n} P(A_k)]^2 = 1$$

then $\Pr(\overline{\lim}\, A_n) = 1$. It is easy to see that the above limit holds if the events A_n are pairwise independent.

The authors, by applying this result, find some new results in the theory of Cantor's series. Their previous results have been published: Rényi, " On the distribution of the digits in Cantor's series " [*Mat. Lapok* (1956) **7**, 77-100], Erdös & Rényi, " Some further statistical properties of the digits in Cantor's series " [*Acta Math. Acad. Sci. Hung.* (1959) **10**, 21-29: abstracted in this present journal, No. 170, **1.5**].

Let $\epsilon_n(x)$, $[x\in(0, 1)]$ be the nth digit of the number x if we represent it in the form of Cantor's series, i.e.,

$$\epsilon_n(x) = [q_n r_{n-1}(x)], \quad r_n(x) = [\![q_n r_{n-1}(x)]\!],$$

where $[t]$ denotes the integral part, $[\![t]\!]$ the fractional part of the real number t and q_n is an arbitrary sequence of integers such that $\sum_{n=1}^{\infty} 1/q_n < \infty$. Let us denote by $\nu_k(x)$ the number of occurrences of the number k in the sequence $\{\epsilon_n(x)\}$. The authors find some results on the order of magnitude of $\nu_k(x)$.

A further result is the following: let $O_N(x)$ denote the number of different numbers in the sequence $\epsilon_1(x), \ldots, \epsilon_N(x)$. Then for almost every x we have

$$\lim_{n\to\infty} D_N(x)/N = 1$$

Putting $R_n = \sum_{j=n}^{\infty} 1/q_j$, the authors show that if $\Sigma\, R_n^{s-1} = +\infty$, but $\Sigma\, R_n^s < +\infty$ for some positive integer S, then there is an infinity of s-tuples of equal digits in the sequence $\epsilon_n(x)$ for almost all x, and s is the greatest number having this property.

Some theorems are proved on the structure of the set of all numbers occurring at least once in the sequence of digits of almost all real numbers x.

(P. Révész)

FRANKX, E. (École Royale Militaire, Bruxelles) **1.5** (1.6)

The strong law of large numbers for uniformly bounded variables—*In Spanish*

Trab. Estadíst. (1959) **10**, 233-237

This paper is a continuation of a former one of the same author [*Trab. Estadíst.* (1958) **9**, 111-115 abstracted in this journal No. 14, **1.6**], in which a criterion, which was called qualitative, for the strong convergence of uniformly bounded variables was given by the condition $|\, x_i - m_{x_j} \,| \leq L$.

In this paper, the matrix of correlations

$$u_{x_i x_j} = \mathscr{E}(x_i - m_{x_i})(x_j - m_{x_j})$$

is used to give the following quantitative criterion: the strong law of large numbers holds for a sequence of uniformly bounded variables x_i if for a large enough n, the condition $S(M_n) = \sum_{i,j=1}^{n} u_{x_i x_j} n^{2-\alpha} K$, where $\alpha > 0$ and K is a constant, is fulfilled. The proof is given by using the qualitative criterion of the first paper.

The special cases of Bernouilli, Poisson and Markoff (multiple chains) are considered as an application of this criterion. In these cases the matrix of correlations is reduced either to the principal diagonal or to a finite number of other parallels to the diagonal with zeros elsewhere in both. To see the validity of the criterion for the case of infinitely many positive correlations for each variable, the author considers a sequence x_i of uniformly bounded variables and the variables deduced from this sequence by the formulas $s_n = \sum_{i=1}^{n} x_i/n$; $t_n = \sum_{1}^{n} s_n/n$. It results from the application of the criterion that the variable t_n converges strongly to $m = \mathscr{E}(x_i)$.

(P. Zoroa)

HLAWKA, E. (Universität Wien) **1.5** (1.0)

Statistics on compact groups, I.—*In German*

Anzeiger der math.-naturw. Österr. Akad. Wiss. (1959) 64-76 (3 references)

Let G denote a compact group (identity e) with enumerable basis and $C(f)$ the vector space of all continuous functions f on G. The author considers positive Radon measures $\mu(f) = \int_G f(x)d\mu$ on G with $\mu(1) = 1$. The convolution of μ_1 and μ_2 is defined by $\mu_1 \star \mu_2 = \iint f(x_1 x_2) d\mu_1(x_1) d\mu_2(x_2)$. This present paper derives limit theorems for sequences of convolutions. For similar work on finite groups, see Worobew [*Mat. Sbornick* (1954) **76**, 89-126] and Böge [*J. reine angew. Math.* (1959) **201**, 150-156, abstracted in this present journal, No. 166, **1.0**].

The author uses as an essential instrument the characteristic functions (= matrices) of each measure μ: see also Kawada & Ito [*Proc. Japan Acad.* (1940) **22**, 977-998]. These characteristic functions are the expectation values (relative to μ) of the irreducible unitary representations (matrices) of the compact group G. The measure μ is said to be regular if the determinants of each characteristic function of μ are different from zero. The expression $\mu'(f) = \int f(x^{-1})d\mu$ is called the conjugate measure of μ. μ is said to be symmetric if $\mu = \mu'$, and μ is called normal if $\mu \star \mu' = \mu' \star \mu$. The following is proved:

(i) μ is symmetric, if and only if each characteristic function is a Hermitian matrix.

(ii) μ is normal, if and only if each characteristic function can be unitarily transformed to diagonal form.

(iii) The eigenvalues of the characteristic functions always have absolute value $\leqq 1$.

A measure ν is called idempotent (stable) if $\nu \star \nu = \nu$. Let μ be an arbitrary measure, ν an idempotent measure which commutes with μ and c a real number; then the Poisson measure is defined by $\pi(\mu, \nu, c) = \nu \star e^{-c} \Sigma \frac{c^k}{k!} \mu^{(k)}$ where $\mu^{(k)}$ is the k-fold convolution of μ with itself. In particular if ν is the measure which assigns the value $f(e)$ to each f of $C(f)$, we write $\pi(\mu, c)$ for short. μ is called infinitely divisible (in the elementary sense) if for every natural number s there exists a measure μ_s with $\mu = \mu_s^{(s)}$. Every Poisson measure is infinitely divisible and it is proved that every symmetric, infinitely divisible measure is the limit of a sequence of symmetric Poisson measures.

Let (μ_{nk}) be a double sequence of symmetric measures ($k = 1, 2, \ldots, k_n$; $n = 1, 2, \ldots$) and let $\mu_n = \mu_{n1} \star \ldots \star \mu_{nk_n}$, $\pi_{nk} = \pi(\mu_{nk}, 1)$, and $\pi_n = \pi_{n1} \star \ldots \star \pi_{nk_n}$. Such a double sequence is called an infinitesimal sequence or i-sequence, if $\lim\limits_{n\to\infty} \operatorname*{Max}\limits_{1 \leq k \leq k_n} | \mu_{nk}(f) - f(e) | = \sigma$ holds for each $f \in C(f)$. It is proved that:

1. For every i-sequence (μ_{nk}) the sequence (μ_n) converges to a regular measure, if and only if this is true for the sequence (π_n) of the corresponding Poisson measures. Both sequences have the same limit.

HLAWKA, E. (Universität, Wien) **1.5** (1.0) *continued*

Statistics on compact groups, I.—*In German*

Anzeiger der math.-naturw. Österr. Akad. Wiss. (1959) 64-76 (3 references)

2. If for an i-sequence (μ_{nk}) $\lim \mu_n$ is a regular measure, then it is infinitely divisible in a certain wider sense.
3. A regular symmetric measure μ is the limit of a sequence (μ_n) of an i-sequence (μ_{nk}) with $\mu_{nk} \star \mu_{nb} = \mu_{nl} \star \mu_{nk}$ for all k, l, n, if and only if μ is infinitely divisible in the elementary sense.

Finally a law of large numbers is proved: For a sequence (μ_{nk}) of symmetric measures the sequence (μ_n) converges to $f(e)$, if and only if this is true for the sequence (π_{nk}).

(W. Uhlmann)

IKEDA, S. (Nihon University, Japan) **1.7** (1.6)

A note on the normal approximation to the sum to independent random variables—*In English*

Ann. Inst. Statist. Math., Tokyo (1959) **11**, 121-130 (8 references, 3 tables)

It is known that the distribution of the standardised sum of independent random variables converges to the normal $N(0, 1)$ if the sequence of the random variables satisfies the Liapunoff condition. In connection with this fact, there is the problem of evaluating the error when we replace the distribution of a finite sum of the random variables by the normal. This problem is called the Liapunoff problem concerning which A. C. Berry gave an evaluation of the error. However, there was a slip in his analysis which was corrected by Takano [*Res. Mem. Inst. Statist. Math.* (1950) **6**, 408-415]. The author of this paper gives an approach to the evaluation of the error which differs from that of Berry and of Takano. That is, he treats the problem by improving the evaluation of the difference between the corresponding characteristic functions which was given by Gnedenko and Kolmogoroff. In the derivation of his result, he first considered a special case where all the random variables have the same mean, variance, and third absolute moment followed by the general case. He also used an inequality given by Berry. To compare the present result with the one given by Berry and Takano, he gives numerical examples for the special case, which show that the present method offers more accurate evaluation of the error when the ratio β/σ is large, where δ^2 and β^3 are the common variance and third absolute moment of the random variables, respectively.

(Y. Suzuki)

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ISII, K. (Inst. Statist. Math., Tokyo) **1.1** (2.0)

Bounds on probability for non-negative random variables—*In English*

Ann. Inst. Statist. Math., Tokyo (1959) **11**, 89-99 (4 references, 6 figures)

This paper is an extension of the author's previous work " On a method for generalisation of Tchebycheff's inequality" [*Ann. Inst. Statist. Math., Tokyo* (1958) **10**, 65-88: abstracted in this journal No. 357 **1.5**] to the case of non-negative random variables. This paper contains some generalisations of the results of Wald.

Suppose that there are given first n moments $\mu_1, \ldots, \mu_n$ of a non-negative random variable X whose distribution is unknown. Let E be a given set contained in the space of non-negative real numbers. The only information on X is that it has the moments $\mu_1, \ldots, \mu_n$. Hence the probability distribution of X is not uniquely determined, and may range over a certain class of distributions. The author studies the method of obtaining the sharp upper and lower bounds of the probability that X will fall on E. The problems of this type which appeared in the literature were restricted to the case of determining the bounds of distribution function $F(x)$ for given x. In this paper the author considers the case that E is any given closed or open set.

It is shown that the bound is obtained by constructing the " extremal distribution " which actually attains the bound, and that the spectrum of the extremal distribution consists of a finite number of points. The method of constructing the extremal distribution can be reduced to solving a finite number of algebraic equations, which might be carried out by a satisfactory computing technique, for example, by using an automatic computer. The number of the equations and that of unknowns are determined by the given moments and the shape of set E. In this manner the problem of obtaining the bound on probability can be reduced to that of solving a finite number of algebraic equations.

Some examples containing Wald's case are considered and the procedure of constructing extremal distributions is illustrated. The proof of the stated facts is briefly outlined at the end of the paper.

(K. Isii)

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JONES, H. L. (University of Chicago) 1.3 (2.6)

How many of a group of random numbers will be usable in selecting a particular sample?—*In English*

J. Amer. Statist. Ass. (1959) **54**, 102-122 (11 references, 4 tables)

When a sample is selected from a finite population by employing random numbers, certain numbers may have to be discarded as not being usable. In the first place, some of the random numbers may not correspond to serial numbers in the population. In the second place, some random numbers may be duplicates of others. The number of usable random numbers remaining is a random variable s with a probability distribution involving parameters T, N, n. Here T denotes the size of the sub-class of individuals that it is desired to sample; n is the size of the group of random numbers initially selected; and N is the size of the universe from which the initial selection is made. Exact formulas for this distribution, and for the factorial moments of the differences between the number remaining and the population size, are derived and discussed. The values $\mathscr{E}(s)$, $\mathscr{E}(s^2)$, and σ_s^2 are obtained as functions of T, N, n. Approximations to the cumulative probability distribution are also suggested, and investigated for special cases. Tables are given showing comparisons of approximations with the actual cumulative distribution of s for selected values of N, n, and T/N. These approximations are of some importance in minimising costs where high-speed computers are used in selecting or generating random numbers, and the initial selection of too many or too few numbers causes trouble and expense. The number of usable random numbers corresponds to the number of occupied cells for some sub-class of an entire class of cells among which balls or other objects are randomly distributed. For the special case where all random numbers in a set correspond to serial numbers in a sample population, the problem of predicting the number of usable random numbers is statistically equivalent to the classical occupancy problem. The problem discussed here is not to be confused with the closely related problem of predicting how many successive random numbers must be selected before the number of usable random numbers agrees with some previously specified number.

(A. Booker)

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KALLIANPUR, G. (Michigan State University, East Lansing) 1.0 (–.–)

A note on perfect probability—*In English*

Ann. Math. Statist. (1959) **30**, 169-172 (6 references)

The purpose of this note is to define and characterise a class of perfect probability spaces. In the sense of Gnedenko & Kolmogoroff, the triplet $(\Omega, \mathscr{F}, \mu)$ is said to be a perfect probability space if μ is probability over the algebra $\mathscr{F}$ of subsets of Ω and if for every univalent, real-valued $\mathscr{F}$-measurable function f the following is true:

For every linear set A, such that $f^{-1}(A) \in \mathscr{F}$, there exists a linear Borel set B with $B \subseteq A$ and

$$\mu\{f^{-1}(B)\} = \mu\{f^{-1}(A)\}.$$

This definition involves the measure μ in an essential manner and leads to the possibility of defining classes of measurable spaces $(\Omega, \mathscr{F})$ with the property that for every probability μ, the space $(\Omega, \mathscr{F}, \mu)$ is perfect.

Blackwell's Lusin spaces are included in the class of spaces considered here, but it is not known whether the latter is, in reality, more general than that of a Lusin space.

(R. L. Anderson)

MILICER-GRUZEWSKA, H. **1.0** (–.–)

Construction of the probability field with the fundamental solution of the normal equation with Hölders coefficient—*In French*

Bull. Acad. Polon. Sci., III (1959) **7**, 673-675

This paper gives the positive answer to the following question: is it possible to construct a probability field in the cartesian product $(R_n \times R_n)$ or two finite dimensional spaces (R_n) using the normed solution $[U(X;\ t,\ Y,\ \tau)$, $K \in R_n,\ 0 \leqq t \leqq \tau]$ of Kolmogoroff's first equation with Hölders coefficients.

(L. Kubik)

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MILICER-GRUZEWSKA, H. **1.0** (–.–)

The probabilistic properties of a parabolic system of equations—*In French*

Bull. Acad. Polon. Sci., III (1959) **7**, 721-728.

The authoress proves that the sums of components of fundamental solutions of a parabolic system of linear equations with partial derivatives of second order and with Hölders coefficients are densities of stochastic probabilities.

(L. Kubik)

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OKAMOTO, M. (Osaka University) **1.6** (2.3)

A convergence theorem for discrete probability distributions—*In English*

Ann. Inst. Statist. Math., Tokyo (1959) **11,** 107-112

In this paper, a convergence theorem is given which is useful for proving that a sequence of real-valued, discrete probability distributions converges to a continuous distribution. It is a modification of Scheffé's " useful " convergence theorem [see Scheffé, H. " A useful convergence theorem for probability distributions " *Ann. Math. Statist.* (1947) **18**, 434-438].

The author states two theorems in both the one-dimension and multi-dimensional case. In the one-dimensional case, for example, the theorem says: let $\{X_n\}(n = 1, 2, \ldots)$ be a sequence of integral valued, discrete random variables and $p_n(r)$ the probability distribution of X_n, $(r = 0, \pm 1, \pm 2, \ldots)$. If there exists a sequence $\{\mu_n\}$ of real numbers and probability density function $f(x)$, such that the sequence $\{\sqrt{n}p_n([\mu_n+\sqrt{n}x])\}$ tends to $f(x)$ for any real x, then $\frac{X_n-\mu_n}{\sqrt{n}}$ converges in law to the random variable with density function $f(x)$. The proof of the theorem is based on the Scheffé's theorem in the above-mentioned paper.

In multi-dimensional case, the theorem is analogous to the one dimensional case, and so its proof is omitted.

As the first example, the author shows by means of his theorem that the hypergeometric distribution is asymptotically normal. The conditional distribution $\{n_{ij}\}$ given the marginal totals $\{n_{i\cdot}, n_{\cdot j}\}$ in a contingency table is rigorously shown to converge to the normal when $n_{i\cdot}$ and $n_{j\cdot}$ tends to infinity. The last example is an application of the theorem in multi-dimensional case. The conditional distribution of $\{n_{ii}, n_{ij}\}$ given the marginal total $n_{i\cdot}(i, j = 1, 2, \ldots, k;\ i<j)$ in the intra-class contingency table is treated.

The author shows that, under the usual limiting condition, the conditional distribution converges to the normal of $\frac{k(k-1)}{2}$ dimensions with mean 0 and the identity variance-covariance matrix.

(Y. Suzuki)

PAGE, E. S. (Durham Univ. Computing Lab., England) **1.3** (3.3)

The distribution of vacancies on a line—*In English*

J. R. Statist. Soc. B. (1959) **21**, 364-374 (4 references, 5 tables)

This paper treats the one-dimensional analogue of the more difficult two-dimensional problems arising out of models of physical absorption of two-atom molecules into a rectangular lattice. Given a line of n points, pairs of adjacent points are removed until only isolated points are left; these are called *vacancies*. The first problem treated is the distribution of the number of these vacancies. Two models (i.e. methods of selecting pairs) are treated. In the first model, any of the $n-1$ adjacent pairs are equally likely to be selected, afterwards any of the $n-4$ remaining pairs (or $n-3$ if an end pair is first chosen) and so on. It is shown here that the chance that the rth point from an end is left isolated is very different for $r = 1, 2, \ldots$. In the second model, at each stage, one point of a putative pair is chosen randomly from the so far unselected points. If this is isolated, it is not removed; if it has just one neighbour, the two are removed as a pair: and if it has two neighbours, one of these is taken randomly to make up the pair and the pair removed. The method of solution for both models is as follows: at each stage a line is broken into two similar lines (one perhaps of zero length) both subjected to the same process; set up the corresponding recurrence relationship for the probability generating function and generate this: solve the resulting differential equation.

The first model yields an explicit solution of the differential equation. Difference equations for the moments are obtained from the basic recurrence and are solved for the first four, the first three moments being explicitly given in the case of the first model (the mean and variance being ne^{-2}, $4ne^{-4}$ to first order). Numerical tables of the β_1, β_2 moment ratios suggests a good approximation to the normal distribution for $n \geqq 20$ and heuristic reasons are given for a normal limit. The generating function is used to derive approximate values for the probabilities of no vacancy and one vacancy. Finally a replacement problem is considered. Here the objects removed are replaced by objects of a different kind giving a set of runs (called " blocks " here) of typical length $2k$. These are then removed in pairs (as were the first kind) and replaced by a third kind so that, in the outcome, there are a number of elements of the second kind trapped in blocks of the third kind. A sampling experiment is reported, tending to show that length is geometrically distributed with a mean numerical volume not very different from the exponential base e. Assuming this geometric form, an expression for the mean number of trapped objects is obtained. The author remarks upon a paper by Jackson & Montroll [*J. Chem. Phys.* (1958) **28**, 1101-1109] where a slightly different model is used for an analysis of the same one-dimensional analogue problem.

(D. E. Barton)

PISTOIA, A. (Inst. of Mathematics, Milan Polytechnic) **1.0** (0.4)
Transforms of distribution functions—*In Italian*
Boll. Un. Mat. Ital. (1959) **14**, 537-542 (4 references)

Let $F(t)$ be any distribution function and $K(x, t)$ a bounded function subject to certain hypotheses of measurability and continuity. The author considers the transforms

$$\psi(x) = \int_{-\infty}^{+\infty} K(x, t)dF(t)$$

of the distribution function and proves that $\psi(x)$ is a characteristic function. He shows that it is of the form

$$\phi(x) = \int_{-\infty}^{+\infty} e^{itx}dF(t)$$

if and only if $K(x, t)$ is a characteristic function for any t real, finite and not belonging to at most a set of null measure.

(T. Salvemini)

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RÉNYI, A. (Math. Inst., Hungarian Academy of Sciences, Budapest) **1.0** (2.8)
Summation methods and probability theory—*In English*
Publ. Math. Inst. Hung. Acad. Sci. (1959) **4**, 389-399 (7 references)

The main purpose of this paper is to give a probabilistic interpretation of summation methods.

In the first part it is shown that the summation method, which consists in forming from a given sequence $\mathscr{S}_k$ ($k = 0, 1, \ldots$) the transformed sequence $t_n = \mathbf{A}\mathscr{S}_n$ defined by $t_n = \sum_{k=0}^{\infty} a_{nk}\mathscr{S}_k$ ($n = 0, 1, \ldots$), where $\mathbf{A} = (a_{nk})$ is an infinite matrix with non-negative elements with row-sums equal to 1 and such that the elements of each column tend to 0, if $n \to +\infty$. The limit of t_n, if it exists, can be interpreted probabilistically: namely, the condition concerning the elements of columns expresses that the random variable ν_n, taking on the non-negative integral value k with probability $a_{nk}(\Pr\{\nu_n = k\} = a_{nk},\ k = 0, 1, \ldots)$ tends in probability to $+\infty$. The mean value of the random term $\mathscr{S}\nu_n$ of the sequence $\mathscr{S}_k$ is shown to be t_n.

In the case of Hausdorff-summation method, that is in the case where

$$a_{nk} = \binom{n}{k} \int_0^1 x^k(1-x)^{n-k}dF(x);\ \nu_n = \beta(n, \xi),$$

where ξ is a random variable in the interval [0, 1] with the distribution function $F(x)$ and $\beta(n, x)$ is a binomially distributed random variable for every fixed value of x, with parameter x. Replacing the binomial distribution by the Poisson distribution, Henriksson's class of methods of summation is obtained. With the aid of this interpretation, and using well-known theorems about the mixture of binomial distributions and Poisson distributions, the author shows how to prove in a simple way several known facts concerning these summation methods.

In the fourth part of the paper the author deals with limiting distributions of divergent series. The $(C, 1)$ limiting distribution of a sequence of real numbers $\{\mathscr{S}_n\}$ is defined by $S(y) = \lim_{n\to+\infty} \frac{1}{n+1} \sum_{\substack{k\leq n\\ s_k<y}} 1$, provided that this limit exists and that $S(y)$ is a distribution function and the convergence on the right takes place for all points of continuity y of $S(y)$. It is also mentioned that the existence of a limiting distribution of a sequence may be considered as the generalisation of almost convergence.

In the last part the author considers the following generalised partial sums of a series

$$\sum_{k=0}^{\infty} a_k = A_n = a_{k_1}+a_{k_2}+\ldots+a_{k_r}$$

if $n = 2^{k_1}+2^{k_2}+\ldots2^{k_r}$ $(k_1>k_2>\ldots>k_r)$. The conjecture that: $\lim_{x\to 1-a} \sum_{k=0}^{\infty} a_k\{x^2/(1+x^{2^k})\} = \mathscr{S}$ results in the series A_n being convergent with sum $2\mathscr{S}$ is proved only with the additional condition that the limit to infinity of a_k is zero.

(K. Bognár)

RÉNYI, A. (Math. Inst., Hungarian Academy of Sciences, Budapest) 1.5 (4.7)

On the central limit theorem for the sum of a random number of independent random variables—*In English*

Acta Math. Acad. Sci. Hung. (1960) **11**, 97-102 (7 references)

Let $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ be a sequence of independent and identically distributed random variables with mean value 0 and variance 1. Let us put $\zeta_n = \xi_1+\xi_2+\ldots+\xi_n$ and $\eta_n = \zeta_n/\sqrt{n}$, then, by the central limit theorem,

$$\lim_{n\to+\infty} \Pr(\eta_n<x) = \phi(x) = 1/\sqrt{2\pi}\int_{-\infty}^{x} e^{-\frac{1}{2}u^2}du(-\infty<x<+\infty).$$

The main result of this paper is the following theorem:

Theorem. If $\nu_n(n = 1, 2, \ldots)$ is a sequence of positive integer-valued random variables such that ν_n/n converges in probability to a positive random variable λ having a discrete distribution, then

$$\lim_{n\to+\infty} \Pr(\eta_{\nu_n}<x) = \phi(x) \quad (-\infty<x<+\infty).$$

It is not assumed that ν_n is independent of the random variable $\xi_k(n, k = 1, 2, \ldots)$.

The proof of the theorem is based on the well-known inequality of Kolmogoroff and on the following lemmas:

Lemma: if τ_n is a sequence of independent random variables such that, putting

$$\tau_n/B_n \sum_{B=1}^{n} \tau_k \quad \text{where} \quad B_n\to+\infty$$

the distribution of the random variable τ_n tends to a limiting distribution, then the conditional distribution of τ_n under any condition having a positive probability tends to the same limiting distribution [see Rényi, " On mixing sequences of sets," *Acta Math. Acad. Sci. Hung.* (1958) **9**, 215-228].

Lemma: If $\nu_n = [n\lambda]$, where λ is a positive random variable having a discrete distribution, then

$$\lim_{n\to+\infty} \Pr(\eta_{\nu_n}<x) = \phi(x).$$

The sign $[x]$ denotes the integral part of the real number x. The author obtains the proof of the theorem by writing

$$\eta_{\nu_n} = \eta_{[\lambda n]}+\sqrt{\frac{[\lambda n]}{\nu_n}}\left(\frac{\zeta_{\nu_n}-\zeta_{[\lambda n]}}{\sqrt{[\lambda n]}}\right)+\frac{\zeta_{[\lambda n]}}{\sqrt{[\lambda n]}}\left(\sqrt{\frac{[\lambda n]}{\nu_n}}-1\right)$$

and verifying that the second and third term of this sum tends to zero in probability as $n\to+\infty$.

(J. Mogyoródi)

THEODORESCU, R. (Institute of Mathematics, Bucharest) 1.0 (1.6)

Some remarks on abstract random variables and functions—*In English and Rumanian*

Bull. Math. Soc. Sci. Math. Phys. R.P.R. (1958) **3**, 343-351

Several notions connected with the concepts of random variable and random function are transposed for the case of random elements and random vectors. Firstly, the notions of moment of second order for a random element with range in a commutative Banach algebra with unit element and of covariance of a random vector of second order are introduced; some properties and conditions of existence for these quantities are then given. Further, stochastic continuity, convergence, differentiability and integrability are extended to random vectors. Finally, analogous problems are considered for properties in mean.

(G. Sîmboan)

THEODORESCU, R. (Institute of Mathematics, Bucharest) **1.0** (0.4)
Remarks on random homomorphisms—*In Rumanian*
Analele Univ. " Parhan " Serie St. Naturii (1959) **22**, 55-58 (5 references)

In this note a probabilised Boolean algebra A is considered and the notion of random homomorphism is introduced, as an application of a given interval in the set of all homomorphisms of B into A, where B is the family of real Borel sets. For these random homomorphisms, a stochastic integral is defined and several of its properties are given.

(G. Sîmboan)

TUCKER, H. G. (University of California, Riverside) **1.5** (10.1)
A generalisation of the Glivenko-Cantelli theorem—*In English*
Ann. Math. Statist. (1959) **30**, 828-830 (3 references)

The theorem to which the author refers states that if $\{X_i\}$ is a sequence of independent, identically distributed random variables with any common distribution function $F(x)$, then the sequence $\{F_n(x)\}$ of empirical distribution functions converges uniformly to $F(x)$ with probability one. The assumption of independence is not necessary. This note proves a generalisation of this theorem in the case where the sequence of random variables is strictly stationary but not necessarily ergodic.

In a subsequent issue of the same journal [*Ann. Math. Statist.* (1959) **30**, 1267-1268] the author reported corrections to certain of the inequalities given in his paper—to numbers 4, 6, 7 and 8—and gave an additional sentence to be inserted after the eighth inequality.

(R. L. Anderson)

VARADARAJAN, V. S. (Indian Statistical Institute, Calcutta) **1.9** (-.-)

On a problem in measure spaces—*In English*

Ann. Math. Statist. (1958) **29**, 1275-1278 (3 references)

In this paper the author proves that the family $\mathscr{F}$ of all random variables is, in a certain sense a closed family. $\mathscr{F}$ is a family of random variables on a probability space Ω which takes values from a separable complete metric space X.

This note makes use of theorems by Larentieff and by Lusin and an example is given to show that, in general, the assumption of completeness of X cannot be avoided.

(W. R. Buckland)

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WESLER, O. (University of Michigan, Ann Arbor) **1.8** (5.1)

Invariance theory and a modified minimax principle—*In English*

Ann. Math. Statist. (1959) **30**, 1-20 (10 references)

For each possible decision procedure ϕ, the statistician is concerned with the values $\rho(\omega, \phi)$ of the risk function as ω ranges over the set Ω of all possible states of nature. The author here deals with the problem of partitioning Ω into sets or slices $\Omega_s (s \in S)$ and devising a decision procedure on the basis of the " sliced-up " risk function $\alpha(s, \phi)$, where

$$\alpha(s, \phi) = \sup_{\omega \in \Omega_s} \rho(\omega, \phi) \quad (s \in S)$$

Decision procedures based on such " sliced-up " risk functions are defined as modified minimax decision procedures. A procedure ϕ is said to be at least as good as another procedure ψ in the modified minimax sense if $\alpha(s, \phi) \leqq \alpha(s, \psi)$ for all $s \in S$. The notions of admissibility and complete class can also be defined in a similar way.

If the problem remains invariant under a group G, the author proposes to slice the parameter space into sets Ω_s which are the orbits under the group G_Ω. Since, for almost all points within an orbit, the risk function remains constant the invariant procedures lose nothing under the modified minimax principle of taking suprema. The author has then proved that, under certain weak assumptions, the class of (almost) invariant decision procedures is essentially complete in the modified minimax sense. To prove this he has used the cluster-point of a sequence of procedures to arrive at a new decision procedure which satisfies the conclusions of the theorem. The Hunt-Stein theorem has also been shown to follow from the above theorem.

The author has presented a game-theoretic model of the modified minimax procedure with the help of a particular kind of mixed game where the space of randomised strategies of the minimax player remains the same in each game. By suitably modifying the probability distribution over the set of games in the light of increasing knowledge, a way has been suggested to bridge the gulf between the Bayes and minimax procedures.

(B. B. Bhattacharyya)

ZOROA, P. (Instituto de Investigaciones Estadísticas, Madrid) **1.1 (–.–)**
The theorem of the probability product—*In Spanish*
Trab. Estadíst. (1959) **10**, 197-208

Let $(\Omega_t, \mathscr{A}_t, \mathscr{B}_t)$, $t \in T$, be probability spaces; there is a probability space $(\Omega, \mathscr{A}, \mathscr{B})$ in which Ω is the space product of the Ω_t and the classes $\mathscr{A}_t$ are independent.

Let us consider the sets $C = A_{t_1} \times A_{t_2} \times \ldots \times A_{t_n T - T_n}$ of measurable rectangles and define the function

$$P_T(C) = \prod_{i=1}^{n} P_{t_i}(A_{t_i}).$$

The class of these measurable rectangles is not a field but the class of its unions in a finite number is. It is possible to extend the function $P_T(.)$ to this field $\mathscr{B}_T$; this function defined over $\mathscr{B}_T$ turns out to be a measure. This is shown by proving that if the monotone sequence B_n is such that $B_n \to \phi$ and $B_n \in \mathscr{B}_T$ then $P(B_n) \to 0$. If $B_n = \Sigma\, C^{ni}$, and C^{ni} are disjoint measurable rectangles, it is necessary previously to carry out a decomposition because of the possibility that the rectangles C^{ni} have not sides over Ω_1 disjoints. For $B_n = \sum_{r=1} A_1^{n(r)} \times B_1^{'nr}$ we have $A_1^{n(r)} \in \Omega$, and $B_1^{'nr} \in \mathscr{B}_1'$, where $\mathscr{B}_1'$ is the field formed from the measurable rectangles of $\prod_{k=1} \Omega_k$. It is then shown that $P_T(B_n) = \Sigma\, P_1(A_1^{N(r)})/P_1'(B_1^{'nr})$.

This decomposition is used to prove that, if B_n converges to the empty set, the correspondent probability $P_T(B_n)$ converges to zero. Hence the function $P_T(.)$ defined over the field $\mathscr{B}_T$ is a measure and it follows from the Extension Theorem that this function may be extended in a unique way to the σ-field $\mathscr{A}_T$ constructed over $\mathscr{B}_T$.

(S. Rios)

ZOROA, P. (Instituto de Investigaciones Estadísticas, Madrid) **1.5 (–.–)**
A sufficient condition for the strong law of large numbers—*In Spanish*
Rev. Acad. Madrid (1959) **53**, 491-494

Let $X_1, X_2, \ldots X_n, \ldots$ be a sequence of random variables with expectations

$$\mathscr{E}(X_i) = \mu_i; \quad i = 1, 2, 3, \ldots$$

If we write $$S_n = \sum_1^n X_i - \sum_1^n \mu_i,$$

the strong law of large numbers holds if

$$\Pr(\operatorname{Lim} S_n/n = 0) = 1$$

is fulfilled. Therefore for every increasing subsequence of the indexes $\{n_i\}$, the probability limit

$$\Pr(\operatorname{Lim} S_n''/n = 0) = 1$$

holds.

The first limit can be obtained reciprocally from the second if the following conditions are fulfilled: for every number $\epsilon > 0$ we can fix a sequence of positive numbers which splits into monotonously increasing parts

$$\alpha_1 \leqq \alpha_2 \leqq \ldots \leqq \alpha_n;\ \alpha_{n+1} \leqq \alpha_{n+2} \leqq \ldots \leqq \alpha_{n_2};\ \ldots$$

such that $\operatorname{Lim\,sup}(n_{i+1}/n_i - 1)\alpha_{n_{i+1}} \leqq k\epsilon$ (where k is independent of ϵ) and $\sum_1^\infty \Pr(|X_n - \mu_n| \geqq \alpha_n) < \alpha$. A special case of this result is the one concerning uniformly bounded variables as studied in an earlier paper by Francks [*Trab. Estadíst.* (1958) **9**, 111-115; abstracted in this present journal, No. 14, **1.6**].

(S. Rios)

AUSTIN, T. L., Jr. **2.1** (0.1)

An approximation to the point of minimum aggregate distance—*In English*

Metron (1959) **19**, 10-21 (10 references, 9 figures)

In connection with several works which appeared in *Metron* in the early and middle 1930's, the author demonstrates an approximate and iterative method for solving the problem of the "point of minimum aggregate distance" or the problem of the "median centre". It should be recalled that the problem is as follows: given a set of fixed points in a plane, locate a new point in the plane having the property that the sum of the absolute distances from it to the original points is less than this sum from any other point.

Let (x_1, y_1), (x_2, y_2), $\ldots(x_n, y_n)$ be n points in the plane. The author states that the coordinates x_0 and y_0 of the median centre are:

$$x_0 = \frac{\sum\limits_{i=1}^{n} \frac{x_i}{[(x_0-x_i)^2+(y_0-y_i)^2]^{\frac{1}{2}}}}{\sum\limits_{i=1}^{n} \frac{1}{[(x_0-x_i)^2+(y_0-y_i)^2]^{\frac{1}{2}}}}; \quad y_0 = \frac{\sum\limits_{i=1}^{m} \frac{y_i}{[(x_0-x_i)^2+(y_0-y_i)^2]^{\frac{1}{2}}}}{\sum\limits_{i=1}^{n} \frac{1}{[(x_0-x_i)^2+(y_0-y_i)^2]^{\frac{1}{2}}}}$$

that is to say: $m = (x_0, y_0)$ is the centroid of a set of points $p_1, p_2, \ldots p_n$ which have weights equal to the reciprocal of their respective distances from m.

The author bases his method principally on the following properties already proved by writers mentioned in the references: if P is one of the given points and mP is the line from the median to P, P may be freely moved along mP any distance (so long as it does not pass through m) without affecting the location of the median.

Hence m is also the centroid of a set of points p_1', p_2', $\ldots p_n'$, lying on a unit circle whose centre is m and such that each line mp_i' contains p_i.

If the set of points $p_1, p_2, \ldots p_n$ is given and we choose a point p_0 in the plane arbitrarily, and if we consider a set of unit vectors:

$$p_0p_i/|\,p_0p_i\,| \quad (i = 1, 2, \ldots n).$$

Where p_0p_i is the vector from p_0 to p_i and $|\,p_0p_i\,|$ is the magnitude, then their resultant is along a line p_0M where M is the centroid of a set of points lying on a unit circle about p_0 at the points where the circle intersects the lines p_0p_i. If p_0 moves, the vectors $p_0p_i/|\,p_0p_i\,|$ will shift direction causing a shift in the location of M relative to p_0. The movement of p_0 and M will continue until the vector resultant is zero, whereupon p_0 and M must coincide.

(C. Benedetti)

565

BAHADUR, R. R. (Teachers College, Columbia University, New York) **2.9** (1.7)

A representation of the joint distribution of responses to n dichotomous items —*In English*

USAF School of Aviation Medicine, Report 59-42 (1959), 26 pages (3 references)

Consider a specified set of n dichotomous items with a joint probability distribution given by p. Let $p_{(1)}$ denote the joint distribution of the n items when they are independent with the same marginal probabilities as under p. Suppose we represent $p = p_{(1)}.f$. An explicit expression is obtained for the correction factor f in terms of the n marginal probabilities and 2^n-n-1 correlation parameters. Certain formal models of dependence, suggested by this expression for f, are defined and discussed. It is pointed out that under certain conditions, the probability distribution of the "total score" throws some light on which model of dependence is appropriate in a given case. A generation of this approach to the case when the items are not necessarily dichotomous is also described.

(R. R. Bahadur)

566

BAHADUR, R. R. (Indian Statistical Institute, Calcutta) **2.4** (1.5)

Some approximations to the binomial distribution function—*In English*

Ann. Math. Statist. (1960) **31**, 43-54 (6 references)

This paper is concerned with the upper tail probabilities associated with the binomial distribution. In particular, the problem considered is to approximate

$$\sum_{r=k}^{n} \binom{n}{r} p^r (1-p)^{n-r},$$

where p is given and $np \leqq k \leqq n$. It is first shown that this form can be written in terms of a hypergeometric function and from this an easily calculated upper bound is developed. The author also gives two somewhat more involved but also more complete methods of bounding the probability. These latter methods are obtained through the representation of the hypergeometric function as a continued fraction. The " goodness " of these bounds is examined and it is shown that the bounds are asymptotically equivalent (as $n \to \infty$) to the probability they approximate if that probability tends to zero.

The normal approximation to the binomial probability is also examined and conditions given which insure their asymptotic equivalence for large n. It is shown that a sufficient condition for this equivalence is that $x = O(n^{1/6})$ where $x = (k-np)/(npq)^{1/2}$; this condition may be weakened for some values of p.

(D. H. Shaffer)

BARBERI, B. (Central Statistical Institute, Rome) **2.0** (0.3)

On the theoretical interpretation of statistical distributions—*In Italian*

Riv. Ital. Econ. Demogr. Statist. (1959) **13**, 225-291 (184 references, 10 tables, 12 figures)

The author starts this paper by deploring the present lack of consideration for studies of empirical statistical distributions and then proposes for these a classification based on the distinctive characters of the elements which constitute the classes represented in a given distribution. Those characters are identified by the author in the *species* and in the *age*—or any other equivalent character based on time—of the said elements, and the classification of the various kinds of distributions is derived through combinations of them. Four types of distributions are thus obtained, corresponding to the well-known Bernoulli, Lexis, Poisson and Coolidge distributions.

The author next examines the problem of the mathematical representation of statistical distributions. Pointing out the two-fold nature (Cartesian-geometric and Galilean-physic) of the concept of function applied to physical and social phenomena, he underlines the fact that for statistical distributions the Galilean aspect of their mathematical representation implies the knowledge of a theoretical model interpreting the distributive phenomenon. For classes of the Bernoullian type, starting from the hypothesis of a normal distribution, he arrives at the following interpretation: " Bernoullian distributions are distributions of individuals about the ' type ' to which they tend to adhere with a force depending, for equal deviations, on a constant characteristic of the ' species '."

More generally, for the choice of the model, the author suggests a few examples taken from the physics of quanta showing in particular the similarity between the Rayleigh-Jeans formula and the Paretian formula for income distributions. He also shows the insufficient force of some well-known models of distributions proposed in the literature; for example, the lognormal for the income distribution and the Pearsonian system of curves.

Following a method first proposed by D'Addario, " *The method of the mean of exceeding values* ", a function is suggested for the distribution of incomes. The values of the function, when given an economic interpretation, express for any given income level x, the number of individuals with income less than x who, in terms of income, are equivalent to one individual with income exceeding x.

(V. Levis)

BARTON, D. E. & DAVID, FLORENCE N. (University College, London) 2.5 (4.0)

Haemocytometer counts and occupancy problems—*In English*

Trab. Estdíst. (1959) **10**, 13-18 (5 references)

In this article the problems of counting haemocytes are studied. These problems also occur in the study of Ecology in connection with the analysis of occupancy problems.

It is supposed that there are N cells and associated with each cell is a random variable which can have either a Poisson or a binomial distribution. Some formulae for the case of homogeneity of the cells are given. The question of possible heterogeneity in the sample populations in each of the cells, presents some interest for applications to Ecology and is also considered by the author.

An estimator of the maximum number of cells which the normal equivalent deviate can cover without overlapping is studied. Lastly an index of David & Moore for composite Poisson distributions is studied: an example on the distribution of fagocytes given by Greenwood & White in *Biometrika* is given.

(P. Zoroa)

COX, D. R. (Birkbeck College, London) 2.5 (8.6)

The analysis of exponentially distributed life-times with two types of failure—*In English*

J. R. Statist. Soc. B (1959) **21**, 411-421 (22 references, 2 tables)

This paper deals with the analysis of failure-time data when there are two, or more, types of failure. An application involving the failure-times of radio transmitters, discussed by Mendenhall & Hader [*Biometrika* (1958) **45**, 504-520: abstracted in this present journal No. 54, **4.3**] is used to illustrate the paper; similar problems arising in fields other than industrial life-testing are briefly mentioned.

Four probability models are discussed:

Model A. Independent Poisson risks. Here failures of Types I and II occur independently in Poisson processes with parameters λ_1, λ_2. The probability that a failure is of type I is $\lambda_1/(\lambda_1+\lambda_2)$, independently of the time at which failure occurs, and the probability density function of failure-time is $(\lambda_1+\lambda_2)e^{-(\lambda_1+\lambda_2)t}$.

Model B. Single risk. Here individuals are of two types, the chance that an individual is of the first type being θ. Individuals of Type I are subject to the first type of failure only, and their probability density function of failure-time is $h_1(t) = \alpha_1 e^{-\alpha_1 t}$, and similarly for individuals of the second type. The probability that a failure is of type I given that it occurs at t is independent of t if and only if $\alpha_1 = \alpha_2$, when the model is identical with model A. There is an immediate generalisation if $h_1(t)$ and $h_2(t)$ are not exponential.

Model C. General independent risks. Here random variables T_1, T_2, defining failure-time of type i, are independent and arbitrarily distributed. The observed time and type of failure are determined by the smaller of T_1, T_2.

Model D. Independent proportional risks. This is the same as Model C, with an additional condition which makes failure-time and type of failure independent.

The general properties of these models are compared. Whether Model B, the single-risk model, is to be used, rather than one of the other models, is to be decided on general grounds.

Some statistical procedures connected with the two-risk models are outlined for dealing with the following problems:

(i) test of whether failure-time and type of failure are independent. If they are dependent, Models A and D are excluded.

(ii) discriminating between Models A and D, especially when there is censoring.

(iii) life-table analysis, under Model D, to test whether one of the component distributions is exponential.

(D. R. Cox)

EDWARDS, A. W. F. (Dept. of Genetics, Cambridge) **2.4** (10.1)
The meaning of binomial distribution—*In English*
Nature (1960) **186**, 1074 (3 references)

In this brief note the author outlines a generalisation of the binomial distribution which resolves some of the difficulties of interpretation associated with the ordinary binomial and its variations due to Lexis and Poisson. This new generalisation is termed the Markoff binomial since it allows for correlation between the probability p in pairs of successive trials.

A matrix of transition probabilities is given together with the appropriate generating function. It is shown that the correlation coefficient between the trials is the determinant of the matrix of transition probabilities. The mean is Np and the variance follows from the variance of the sum of N correlated variables. It will exceed the ordinary binomial variance if the correlation is positive and vice versa.

The author states that in practice it is necessary to be able to express the data in series form. Useful inferences cannot be made from the fact that data conform to a simple binomial since both variability of p between experiments and positive correlation between successive trials will increase the variance: and vice versa.

(W. R. Buckland)

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GEORGESCU-ROEGEN, N. (Vanderbilt University, Tennessee) **2.1** (-.-)
On the extreme of some statistical coefficients—*In English*
Metron (1959) **19**, 38-45 (1 reference)

In this article, the author suggests a uniform method for maximising or minimising some homogeneous algebraic forms in particular domains. These algebracic forms include several statistical coefficients (correlation coefficients, Pearsonian coefficients, etc.). The author gives first an alternative proof of some of Benedetti's theorems about algebraic forms, including correlation coefficients and dissimilarity coefficients, and secondly establishes two new theorems on Pearsonian coefficients. The two theorems of Benedetti which are proved are as follows:

(i) let two sets of n real variables (x_i) and (y_i) satisfy the conditions

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i = 0; \quad \sum_{i=1}^{n} x_i^2 = \sum_{i=1}^{n} y_i^2 = 1$$

where $x_i \leqq x_{i+1}$; $y_i \leqq y_{i+1}$ $(i = 1, 2, \ldots n-1)$

then $$\min \sum_{i=1}^{n} x_i y_i = 1/(n-1).$$

(ii) let two sets of n variables (x_i) and (y_i) satisfy the conditions

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i = 0; \quad \sum_{i=1}^{n} | x_i | = \sum_{i=1}^{n} | y_i | = 1$$

where $x_i \leqq x_{i+1}$; $y_i \leqq y_{i+1}$ $(i = 1, 2, \ldots n-1)$

then $$\max \sum_{i=1}^{n} | x_i - y_i | = 2(n-2)/(n-1).$$

The two new theorems proved by the author are:

(*a*) let the set (x_i) of n real variables satisfy the following conditions

$$\sum_{i=1}^{n} x_i = 0; \quad \sum_{i=1}^{n} x_i^2 = 1$$

if $S_k = \sum_{i=1}^{n} x_i^k$ where k is a positive integer, then

$$\max | S_{2j+1} | = (n-1)^{2j} - 1/\sqrt{n^{2j-1}(n-1)^{2j-1}}$$

(*b*) assuming the same conditions as in theorem (*a*), we have

$$\max S_{2j} = (n-1)^{2j-1} + 1/n^j(n-1)^{j-1}$$

$$\min S_{2j} = \begin{cases} 1/n^j[\{(\nu+1)^j/\nu^{j-1}\} + \{\nu^j/(\nu+1)^{j-1}\}] & \text{if } n = 2\nu+1 \\ 1/n^{j-1} & \text{if } n = 2\nu \end{cases}$$

For the Pearsonian coefficients there is the corollary that

$$\max | \beta_{2j+1} | = (n-1)^{2j} - 1/n\sqrt{(n-1)^{2j-1}};$$
$$\max \beta_{2j} = (n-1)^{2j-1}/n(n-1)^{j-1}$$

$$\min \beta_{2j} = \begin{cases} 1/n[\{(\nu+1)^j/\nu^{j-1}\} + \{\nu^j/(\nu+1)^{j-1}\}] & \text{if } n = 2\nu+1 \\ 1 & \text{if } n = 2\nu \end{cases}$$

These coefficients are here defined as $\beta_k = \mu_k/\mu_2^{k/2}$ where μ_k is the moment of kth order about the mean. The methods used by the author in the proofs of the theorems include notions of topology.

572

(C. Benedetti)

GIARDINA, B. (University of Turin) **2.1** (11.0)

Maximum relative error in the calculation of the arithmetic mean and of the variance of data grouped in frequency distributions and a criterion for the determination of the number of classes—*In Italian*

Boll. Cent. Ric. Operat. (1959) **3**, 29-33 (1 table)

Let n elements be grouped in classes, of a constant size a and $\bar{x}$ be their arithmetic average, while $\bar{x}'$ is the average obtained using the central values of the classes. The author shows that

$$|\bar{x}-\bar{x}'| \leqq \frac{a}{2}$$

and therefore the maximum relative error is $a/2\bar{x}'$.

Analogous results are reached in the case of the variance, it being shown that in this case the maximum relative error equals $2a\bar{x}'/s'^2$ ($\bar{x}'>0$ by hypothesis), where s'^2 is the statistic obtained making use of the central values of the classes.

Starting from the above formulae one can determine the size of the classes such that one or the other of the stated maximum relative errors, or both, do not exceed prescribed limits.

An example illustrates the theory.

(C. Grossi)

573

GINI, C. & SONNINO, G. (University of Rome) **2.1** (–.–)

Some simple formulæ useful for the median and dividing values—*In Italian*

Metron (1959) **19**, 46-91 (3 references, 14 tables)

In this paper the authors establish sound relations for the determination of the median value, of the " dividing values " and, in general, of the tantiles in frequency distributions grouped into classes.

These relations are obtained by fitting linear and hyperbolic functions to represent the density of frequency in the class-intervals.

The parameters of $f(x) = mx+n$ or $(m/x)+n = f(x)$ are determined by the conditions satisfied in the generic class-interval (a, b):

$$\int_a^b f(x)dx = N \quad \text{effective class-frequency in } (a, b)$$

$$\int_a^b xf(x)dx = A \quad \text{effective moment in } (a, b).$$

There are several numerical applications in the field of income distributions.

(C. Benedetti)

HERZEL, A. (Faculty of Statistics, University of Rome) 2.1 (–.–)

On the determination of the median values of distribution—*In Italian*

Metron (1959) **19**, 92-109 (3 references, 1 table)

The problem treated in this article is the following:

> Let a frequency distribution be obtained by the grouping into classes of a continuous distribution. What is the median value (for the maximum information about it) of the unknown original distribution?

This problem is analogous to the problem of corrections to moments and the author approaches it under the hypothesis that the original distribution curve is a generic polygon with every vertex in correspondence to the class extrema.

The author establishes the interval where the median values must fall under the polygonal hypothesis. The median value of the original distribution is also sought under more general conditions about the mathematical form of the unknown distribution.

The results obtained are also used for determining the percentiles.

(C. Benedetti)

575

KHATRI, C. G. (University of Baroda) 2.1 (2.5)

On certain properties of power-series distributions—*In English*

Biometrika (1959) **46**, 486-490 (1 reference)

Starting from $\Pr\{\xi = x\} = a_x Z^x/A$, $x = 0, 1, 2, \ldots$, where the numerator is defined as non-negative, a is a function of x, or constant and A is the sum of the numerator over all possible x, the author establishes recurrence relations between the cumulants and the factorial-cumulants of ξ. He uses these recurrence values to demonstrate that any power-series distribution involving a single parameter is determined uniquely from its first two moments.

Illustrative examples given include the negative binomial distribution of one parameter, the truncated negative binomial distribution, the positive binomial distribution and the Poisson distribution. A multivariate extension of the theory is given. The multinomial distribution is used to illustrate this.

(Florence N. David)

576

LINNIK, Y. V. (Leningrad) **2.1** (2.3)

General theorems on the factorisation of infinitely divisible laws—*In Russian*

Teor. Veroyat. Primen. (1959) **4**, 55-85 (7 references)

This article is the second part of an extensive study [the first part was published in *Teor. Veroyat. Primen.* (1958) **3**, 3-40]. In this part the detailed proof of sufficiency of the conditions of the following theorem is given:

For an infinitely divisible law with a Gaussian component $\gamma > 0$ to decompose only into infinitely divisible components it is necessary that its Poissonian spectrum be finite or countable. That is, the logarithm of its characteristic function is

$$\log \phi(t) = i\beta t - \gamma t^2 + \sum_{n=1}^{\infty}\left[\lambda_n\left(e^{-it\mu_n} - 1 - \frac{it\mu_n}{1+\mu_n^2}\right) + \lambda_{-n}\left(e^{it\nu_n} - 1 - \frac{it\nu_n}{1+\nu_n^2}\right)\right]$$

Moreover, the corresponding Poissonian frequences μ_n and ν_n must coincide with the following series of numbers:

for μ_n

$$\ldots\ k_{-1}k_{-2}\mu,\ k_{-1}\mu,\ \mu,\ k_1^{-1}\mu,\ k_1^{-1}k_2^{-1}\mu,\ \ldots$$

for ν_n

$$\ldots\ l_{-1}l_{-2}\nu,\ l_{-1}\nu,\ \nu,\ l_1^{-1}\nu,\ l_1^{-1}l_2^{-1}\nu,\ \ldots$$

where $\ldots k_{-2}, k_{-1}, k_1, \ldots$ and $\ldots l_{-2}, l_{-1}, l_1, \ldots$ are arbitrary sets of natural numbers >1. If the Poissonian spectrum is bounded, this condition is also sufficient.

(B. V. Gnedenko).

577

LORD, F. M. (Educational Testing Service, Princeton) **2.1** (2.8)

An empirical study of the normality and independence of errors of measurement in test scores—*In English*

Psychometrika (1960) **25**, 91-104 (7 references, 5 tables)

The author points out that it is usually assumed that errors of measurement are distributed normally, independently of each other and of the true value as being measured. In most mental testing situations, however, there are upper and lower limits to the range of scores that may be assigned. It follows from this that the frequency distribution of the errors of measurement cannot be the same when the true score is close to zero, for example, as it is when the true score has some less extreme value. This paper represents an attempt to determine to what extent deviations from these assumptions occur in practice. The author shows that the third-order moments of true scores and errors of measurement in terms of the observable moments of x_u and x_v can be used to test the hypotheses of the normality and independence of the errors of measurement.

Data obtained from the administration of the 150 item vocabulary test to a nationwide sample of about 13,000 college seniors was used to determine the extent to which the assumptions were tenable. From this data four groups of 1000 examinees each were selected; the group of examinees with the lowest observed scores, a group with middle scores, a group with the highest scores, and a spaced sample of all remaining examinees. It was found that for all four groups the association between true scores and errors of measurement was much larger than could be plausibly accounted for under the hypothesis of independence and normality. In three of the four groups the skewness of the error distribution differs from zero by more than could be plausibly accounted for. The skewness of the errors of measurement is greatest for the most extreme groups, as would be expected *a priori*. Additional numerical results suggest that in the present data the standard error of measurement tends to be slightly larger for low true scores than for high true scores.

The mathematical rationale underlying the statistical treatment of the problem is presented and standard error formulas are given for making necessary significance tests. Two tables are presented giving the coefficients and formulas for the sampling variances and covariances of the third-order multivariate k-statistics of the observed scores when the tests are parallel and the errors of measurement are normally distributed independently of true scores; and for coefficients and the formulas for the sampling variances and covariances of the best estimates of the moments of the observed scores and true score and error scores under the same conditions.

(R. E. Stoltz)

MALLOWS, C. L. (University College, London) **2.2** (0.6)

On the probability integral transformation—*In English*

Biometrika (1959) **46**, 481-483 (5 references)

The probability integral transformation can be used to replace a set of n independent random variables with a common known distribution by a set of n independent uniform variables. If the common distribution is not completely known, but is of known form with s unknown parameters which are estimated from the sample, then the transformation gives variables which in general are neither uniform nor independent. The resulting joint distribution may depend on the form of the original distribution, and even on the values of the parameters.

In this present note, it is shown that a phenomenon observed by David & Johnson [*Biometrika* (1948) **35**, 182-190], namely, the existence of functional constraints between the transformed variables, is not at all general. This loss of dimensionality (when it exists) does not correspond to the asymptotic partial loss of degrees of freedom of certain χ^2 and ψ^2 test criteria.

The Jacobian $\mathbf{J}$ of the transformation is written down, and is related to a certain $(s \times s)$ matrix. Some cases in which this matrix has zero rank, so that the rank of $\mathbf{J}$ is $n-s$, are investigated. For $s = 1$ and 2, this occurs in translation parameter problems as well as problems involving both translation and scale parameters. In general, the relations between the estimates and the distributions which imply a reduction of dimensionability do not seem to have anything to do with maximum likelihood.

(C. L. Mallows)

RÉNYI, A. (Math. Inst., Hungarian Academy of Sciences, Budapest) **2.9** (1.5)

On connected graphs, I.—*In English*

Publ. Math. Inst. Hung. Acad. Sci. (1960) **4**, 385-388 (6 references, 1 figure)

The author investigates the connected graphs consisting of n vertices and n edges, that is the connected graphs containing exactly one circle. The number of such graphs is known to be

$$C(n, n) = \tfrac{1}{2} n^{n-1} \sum_{k=3}^{n} \binom{n}{k}^{k!/n^k} \sim \sqrt{\pi/8}\, n^{n-\frac{1}{2}}$$

—see L. Katz, "Probability of indecomposability of a random mapping function" [*Ann. Math. Statist.* (1955) **26**, 512-517]. The distribution of the length of the circle is determined.

If γ_n denotes the length of the circle contained in a graph chosen at random, so that each of the $C(n, n)$ graphs has the same probability to be chosen, then

$$\Pr\{\gamma_n < x\} = (n^{n-1}) \sum_{3 \leqq k < x} \prod_{j=1}^{k-1} (1 - j/n)$$

and $\gamma_n/\sqrt{n}$ has in the limit for $n \to +\infty$ the same distribution as the absolute value of a standardised normal variate. It follows that the mean value of γ_n is asymptotically $\sqrt{2n/\pi}$.

(K. Sarkadi)

TSAREGRADSKI, I. P. (Moscow University) **2.4** (2.2)

On a uniform approximation for a binomial distribution of infinitely divisible laws—*In Russian*

Teor. Veroyat. Primen. (1958) **3**, 470-474 (2 references)

Let $\mathscr{F}_p^n(x)$ be (n, p)-binomial distribution function and be the set of all infinitely divisible laws. We define

$$\rho(\mathscr{F}_p^n, \mathscr{L}) = \inf_{G \in \mathscr{L}} \sup | \mathscr{F}_p^n(x) - G(x) |,$$

then

$$\sup_{0 \leq p \leq 1} \rho(\mathscr{F}_p^n, \mathscr{L}) \leqq \frac{C_0}{\sqrt{n}},$$

where C_0 is an absolute constant.

(I. P. Tsaregradski)

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WEILER, H. (Commonw. Sci. Industr. Res. Org., Prospect, N.S.W., Australia) **2.8** (11.1)

Means and standard deviation of a truncated normal bivariate distribution—*In English*

Aust. J. Statist. (1959) **1**, 73-81 (4 references, 1 table, 5 figures)

In this paper, the author is concerned with the standardised normal bivariate distribution $\phi(x, y;\ \rho)$ with correlation coefficient ρ which is truncated within a region $D\{a \leqq x \leqq \infty, b \leqq y < \infty\}$. The truncated probability distribution is $\phi(x, y;\ \rho)/\Pr(a, b;\ \rho)$ where

$$\Pr(a, b;\ \rho) = \iint_D \phi(x, y;\ \rho) dx\, dy$$

and the truncated moments are given by

$$\mathscr{E}(x^s t^t) = \{\Pr(a, b;\ \rho)\}^{-1} \iint_D x^s y^t \phi(x, y;\ \rho) dx\, dy.$$

Various formulae are established relating a, b and $\Pr(a, b;\ \rho)$ with the first and second moments, $\mathscr{E}(x)$, $\mathscr{E}(y)$, $\mathscr{E}(x^2)$, $\mathscr{E}(y^2)$, $\mathscr{E}(xy)$.

Making use of Pearson's Tables for $\Pr(a, b;\ \rho)$ [Pearson, K., *Tables for Statisticians and Biometricians*, Part I, II (1930), (1931) Cambridge] and the established relationships, a table for the standard deviations of the truncated distribution is obtained. Charts giving $\mathscr{E}(x)$ and $\mathscr{E}(y)$ in terms of a, b are constructed for several sets of a, b and ρ, including the case of $b = -\infty$, that is to say, where there is truncation on x alone.

(B. Weesakul)

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BENNETT, J. F. & HAYS, W. L. (Lincoln Lab., Massachusetts Inst. Tech. and Univ. Michigan, Ann Arbor) **3.6** (3.8)
Multi-dimensional unfolding: determining the dimensionality of ranked preference data—*In English*
Psychometrika (1960) **25**, 27-43 (7 references, 1 table, 6 figures)

A model is proposed which treats rankings given by a group of judges as representing regions in an isotonic space of dimensionality r. The model proposed is simply a multi-dimensional generalisation of Coombs' method of unfolding. The lengthy discussion of the model and a comparison of three methods of determining the dimensionality of the space are given: dimension by mutual boundary, dimension by cardinality, and dimensions by groups. The latter two methods are discussed in detail. The cardinality method imposes a lower bound on the possible dimension of the system by a comparison of the total number of different rankings returned by the judges, with $C(n, k)$, the maximum possible number of elemental isotonic regions that can be generated by n objects in k dimensions. It is shown that $C(n, k) = C(n-1, k) = (n-1)C(n-1, k-1)$. The authors point out that no satisfactory non-recursive expression for this equation has been found. However, the identity of the values obtained from it with sums of absolute values of Stirling numbers of the first kind are given in the table for several combinations of objects and dimensions.

The dimension by groups criterion is based on the notion that the minimum dimension of the space in which a complete solution may be realised must be one less than the number of elements in the largest transposition group present in the experimental data. Of the three criteria for dimensionality, the groups criterion appears to be the most practical and to a certain extent the most sensitive for use with preference data. The authors suggest a modification of this criterion for use with more limited data, and also suggest a very stringent criterion obtained by a combination of the cardinality and the groups criteria.

(R. E. Stoltz)

BONEAU, C. A. (Duke University, North Carolina) **3.1** (7.5)
The effects of violations of assumptions underlying the t-test—*In English*
Psychological Bulletin (1960) **57**, 49-63 (22 references, 1 table, 10 figures)

Random samples were drawn from populations which were either normal, rectangular, or exponential in distribution, with means equal to zero and variances of 1 or 4. For several combinations of forms and variances, t-tests of the significance of the difference between sample means were computed using combinations of the various conditions, using sample sizes of 5 and 15. For each of these combinations frequency distributions of sample t's were obtained by use of an electric computor. The values of t obtained thereby were compared with the theoretical values. The author concludes that for a large number of situations confronting the researcher, the use of the ordinary t-test and its associated tables will result in probability statements which are accurate to a high degree even though the assumptions of homogeneity of variance and normality of the underlying distribution are untenable.

This large number of situations have the following general characteristics: (i) the two sample sizes are equal or nearly so, (ii) the assumed underlying population distributions are of the same shape or nearly so. If the distributions are skewed they should have nearly the same variance. If the sample sizes are unequal, one is in no difficulty providing the variances are compensatingly equal. The combination of unequal sample sizes and unequal variances automatically produces inaccurate probability statements, which can be quite different from the nominal values. If the two underlying populations are not the same, there seems to be little difficulty if the distributions are both symmetrical. Increasing the sample size has the effect of removing the skewness, and the normal distribution is approached.

The author concludes that with sample sizes of 25 or greater, the t-test seems to be functionally distribution-free. He also points out that since the t and F tests of analysis of variance are intimately related, it can be shown that many of the statements referring to the t test can be generalised quite readily to the F test.

(R. E. Stoltz)

BOSE, R. C. & GUPTA, S. S. (University of North Carolina, Chapel Hill) **3.8** (2.3)
Moments of order statistics from a normal population—*In English*
Biometrika (1959) **46**, 433-440 (22 references, 2 tables)

Suppose the kth order statistic in a sample of size n from a unit normal population to be written $X(k)$. This paper gives the moments of $X(k)$, which have been given, and are acknowledged to have been given, many times before. A new technique for their derivation is set out which uses some of the integrals previously calculated by Hojo [*Biometrika* (1931) **23**, 315-360] when discussing the same problem. Exact values are deduced for such cases where they exist. Numerical illustrations are given for the third crude moment of $X(k)$ for $n = 2, 3, 4, 5$; $k = n, n-1, n-2$; and for the fourth crude moment of $X(k)$ for $n = 2, 3, 4, 5, 6$; $k = n, n-1, n-2$.

(Florence N. David)

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BUSH, K. A. & OLKIN, I. (University of Idaho and Michigan State University) **3.4** (8.1)
Extrema of quadratic forms with applications to statistics—*In English*
Biometrika (1959) **46**, 483-486 (1 reference)

Two sharp inequalities for quadratic forms are given by the authors in this paper. The first is for the minimum of a positive definite quadratic form subject to a set of linear constraints, and the second is for the maximum of the ratio of a positive semi-definite form to positive definite form.

These inequalities are applied to four statistical problems where the quadratic forms are subject to certain restrictions:

(i) Stratified sampling. The problem is that of optimum allocation of the sample to the strata.

(ii) Moments. The problem is that of the minimum distance of a point in k-dimensional space from the origin where the sum of $i^r w_i$ is given as unity for $r = 0, 1, \ldots, m$; $i = 1, 2, \ldots, k$.

(iii) The choice of the weighting vector to give the maximum discrimination.

(iv) The choice of the weight functions to maximise the correlation between two linear combinations of the observations.

(Florence N. David)

ROGERS, G. S. (University of Arizona, Tuscon) 3.8 (-.-)

A note of the stochastic independence of functions of order statistics—*In English*

Ann. Math. Statist. (1959) **30**, 1263-1264 (2 references)

In this note the author uses a particular form of a well-known result to prove that the following two statements are equivalent:

(i) z is the independent of x_k for some $k \geqq j$

(ii) z is independent of the set $\{x_j, x_{j+1}, \ldots x_n\}$.

This is done on the basis of an ordered sample of n random observations, from a population with a non-degenerate probability density function, $(x, \leqq n_2 \leqq \ldots \leqq x_n)$, and z a statistic based upon $(x_1, x_2, \ldots x_j)$.

The author states that it remains an open question whether this note can be extended to discrete or mixed cases.

(R. L. Anderson)

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RUBEN, H. (Columbia University, New York) 3.3 (-.-)

On the distribution of the weighted differences of two independent Student variables—*In English*

J. R. Statist. Soc. B (1960) **22**, 188-194 (14 references)

If t_1 and t_2 are two independent Student variables with degrees of freedom f_1 and f_2 respectively and if θ is a fixed angle between 0 and $\pi/2$, then the d statistic is defined as $t_1 \sin \theta - t_2 \cos \theta$. As proposed by Fisher, $\tan \theta$ is equal to the ratio of the estimated standard errors of the sample means in the Fisher-Berens problem of the difference between the means of two normally distributed samples.

The author introduces as auxiliary variables ξ_1, ξ_2, u_1 and u_2 where ξ_1 and ξ_2 are independent unit normal variables and u_1 and u_2 are independent chi-squared variables with degrees of freedom f_1 and f_2 respectively. He writes t_1 as the ratio ξ_1 over the root of u_1/f_1 and similarly for t_2 and calls v, the variant of the d-statistic on substitution for t_1 and t_2. The conditional probability density function of v given u_1 and u_2 and the unconditional probability density function of v are obtained. By manipulating the unconditional probability density function it is shown that it can be reduced to the integral over the range (0, 1) of a beta function weighted by an incomplete beta function ratio. It is suggested that this form is a useful one for computing the distribution of v.

The case $f_1 = f_2 = f$, $\theta = \pi/4$ and the case $f_1 = \infty$ are discussed in detail.

(Florence N. David)

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SHENTON, L. R. (College of Science & Technology, Manchester) **3.6** (4.1)
The distribution of moment estimators—*In English*
Biometrika (1959) **46**, 296-305 (7 references, 3 figures, 1 table)

In a previous paper the author introduced a class of moment estimators dependent on the sample moments [*Biometrika* (1958) **45**, 311-320: abstracted in this present journal, No. 58, **4.1**]. He now proceeds to consider the distribution of a moment estimator.

The first four cumulants of the estimator are calculated and the suggestion is made that the distribution is asymptotically normal. It is shown that, where the moment estimator depends on an infinity of moments, the cumulants are the same as those of the maximum likelihood estimator if it exists.

The case of a single parameter only is treated. The question of the simultaneous estimator of several parameters is deferred for further consideration.

If x is the random variable, continuous, possessing all moments and with range independent of θ the parameter to be estimated, the author considers an artificial distribution function used in his earlier (cited) paper to illustrate his present research. The artificial distribution function referred to above is $\exp(-x)$ multiplied by a function linear in x and θ.

(Florence N. David)

WALSH, J. E. (System Develop. Corp., Santa Monica, California) **3.8** (4.1)
Non-parametric properties of some maximum likelihood estimates for median of symmetrical population—*In English*
Ann. Inst. Statist. Math., Tokyo (1959) **11**, 81-88 (1 reference)

Let $x(1)<\ldots<x(n)$ be the order statistics of a sample of size n from a symmetrical population with median ϕ. The author shows that unique maximum likelihood estimates which are based on specified sets of the order statistics can be obtained for ϕ, if the population is of a given parametric form and has a frequency function sufficiently well-behaved. He also shows that the maximum likelihood estimate of ϕ has non-parametric properties, when the set used consists of pairs of the type $x(i)$ and $x(n+1-i)$, perhaps also $x[(n+1)/2]$ if n is an odd number.

Five assumptions denoted from (*a*) to (*e*), are required in order to get the main result (theorem 2) of this paper. Assumption (*a*) requires un-tied observations, and (*b*) is the assumption that permissible data for use in estimating ϕ contain one or more different pairs of the form $x(i)$, $x(n+1-i)$. The assumptions (*c*), (*d*) and (*e*) relate to the frequency function.

Let $(\hat{\phi}, \hat{\theta}_1, \ldots, \hat{\theta}_k)$ be the values for the set of population parameters $(\phi, \theta_1, \ldots, \theta_k)$ that maximise the joint distribution of order statistics which contain different pairs of the form $x(i)$, $x(n+1-i)$; then $\hat{\phi} = \hat{\phi}[x(i_1), x(i_2), \ldots, x(n+1-i_2), x(n+1-i_1)]$ is the maximum likelihood estimate for ϕ.

The main result of this paper is that the statistic $\hat{\phi}$ is symmetrically distributed about ϕ.

(H. Hudimoto)

WEILER, H. (Commonw. Sci. Industr. Res. Org., Prospect, N.S.W., Australia) **3.6** (4.1)

Note on harmonic and geometric means—*In English*

Aust. J. Statist. (1959) **1**, 44-46 (1 reference)

It frequently happens in statistical problems that the harmonic mean $H = [1/n\{\Sigma(x_i^{-1})\}]^{-1}$ of a sample of n positive numbers $x_1, x_2, \ldots, x_n$ is involved instead of the arithmetic mean $\bar{x} = 1/n\{\Sigma(x_i)\}$, the harmonic mean being an unbiased estimate of some population parameter. However, when large samples are involved, it is usually simpler to calculate arithmetic means. In this paper the author deals with the error introduced in using the arithmetic mean instead of the harmonic mean. To this effect, he proves the following theorem:

if $a>0$ is the smallest and $b \geqq a$ the largest of the values $x_1, x_2, \ldots, x_n$, then

$$0 \leqq (\bar{x}-H)/H \leqq (b-a)^2/4ab$$

where the first equality holds only when all values x_i are equal and the second only when one-half of the values are equal to a and the other half equal to b.

The above inequality shows that $\bar{x}$ differs little from H provided $(b-a)$ is small compared with a. Finally since the geometric mean lies between the arithmetic and the harmonic mean, the inequality continues to hold when the harmonic mean is replaced by the geometric mean.

(R. M. Phatarfod)

WHITTLE, P. (Statistical Laboratory, Cambridge) **3.4** (2.5)

Quadratic forms in Poisson and multinomial variables—*In English*

J. Aust. Math. Soc. (1960) **1**, 233-240 (4 references)

This paper is concerned with the lower bounds for the distribution function of positive definite quadratic forms in independent Poisson variables, or a set of multinomial variables. The following two theorems are proved:—

Theorem 1: If n_j ($j = 1, 2, \ldots, m$) denote a set of independent Poisson variables with $\mathscr{E}(n_j) = \lambda_j$, and with deviations $\Delta_j = n_j - \lambda_j$ from expectation, then for a positive definite matrix $\mathbf{H} = \{h_{ij}\}$,

$$\Pr\{\mathbf{\Delta}'\mathbf{H}\mathbf{\Delta} \leqq S\} > \prod_j \left\{\frac{\exp(-S/24\lambda_j g)}{\{1+(S/\lambda_j g)^{\frac{1}{2}}\}^{\frac{1}{2}}}\right\} \Pr\{\mathbf{X}'\mathbf{H}\mathbf{X} \leqq \mu S,$$

provided $mh/4 \leqq S \leqq g \min\limits_j \lambda_j$; where h is the greatest eigenvalue of $\mathbf{H}$, g the least eigenvalue of $\mathbf{G} = \{h_{jk}(\lambda_j\lambda_k)^{\frac{1}{2}}\}$, $\mu = \{1-(mh/4S)^{\frac{1}{2}}\}^2$, and $\mathbf{X}$ has a multivariate normal distribution with mean $\mathscr{E}(\mathbf{X}) = \mathscr{E}(\mathbf{\Delta}) = 0$ and covariance matrix $\mathscr{E}(\mathbf{XX}') = \mathscr{E}(\mathbf{\Delta\Delta}') = \{\delta_{ji}\lambda_j\}$.

Theorem 2: If n_j($j = 1, 2, \ldots, m$) denote a set of multinomial variables with $\mathscr{E}(n_j) = Np_j$, where $\Sigma p_j = 1$, and with deviations $\Delta_j = n_j - Np_j$ from expectation, then for a positive definite matrix $\mathbf{H} = \{h_{ij}\}$,

$$\Pr\{\mathbf{\Delta}'\mathbf{H}\mathbf{\Delta} \leqq S\} > \frac{\exp -(mS/8Ng \min p_j)}{\prod_j \{1+(S/Ngp_j)^{\frac{1}{2}}\}^{\frac{1}{2}}} \Pr\{\mathbf{X}'\mathbf{H}\mathbf{X} \leqq \mu S\},$$

provided $mh/4 \leqq S \leqq Ng \min\limits_j p_j$; where h is the greatest eigenvalue of $\mathbf{H}$, g the least eigenvalue of $\mathbf{G} = \{Nh_{jk}(p_jp_k)^{\frac{1}{2}}\}$, $\mu = \{1-(mh/4S)^{\frac{1}{2}}\}^2$, and $\mathbf{X}$ has a multivariate normal distribution with mean $\mathscr{E}(\mathbf{X}) = \mathscr{E}(\mathbf{\Delta}) = 0$ and covariance matrix $\mathscr{E}(\mathbf{XX}') = \mathscr{E}(\mathbf{\Delta\Delta}') = N\{p_j\delta_{ji} - p_jp_i\}$.

(J. Gani)

BEALE, E. M. L. (Admiralty Research Labs., Teddington, England) **4.2** (4.4)

Confidence regions in non-linear estimation—*In English*

J. R. Statist. Soc. B (1960) **22**, 41-76 (13 references, 2 tables)

The author supposes the means of a set of n normally distributed observations with common variance σ^2 are known non-linear functions of p unknown parameters. A natural approximate confidence region for the parameters is obtained by setting an upper bound (calculated from a chi-squared or F distribution) to the excess of the residual sum of squares over its minimum value. This reduces to the standard confidence region when the hypothesis is linear, or is linear in some transformed set of parameters. In this paper some properties of this " likelihood region " are investigated for moderately non-linear situations, and it is compared with the exact region proposed by Wilks & Daly [*Ann. Math. Statist.* (1939) **10**, 225-235].

The solution locus in the sample space is the p-dimensional set of points for which the model can be fitted exactly. It is assumed that this is differentiable up to the third order. It is shown that if a uniform prior distribution over the solution locus is assumed, then the posterior distribution is approximately normal with a covariance matrix which depends on the quadratic terms in the expression for the solution locus.

Turning now to the confidence region argument, a second-order approximation is given for the probability content of the acceptance region corresponding to any chosen parameter point; that is the region in the sample space containing all points for which this parameter point will be accepted. The term giving the departure from the nominal confidence coefficient involves a certain measure of non-linearity, which has a fairly simple interpretation. The significance of this can be determined simply, and its estimation is discussed.

The author shows that, in a certain " isotropic " case in which part of the specification of the sample has the role of an ancillary statistic, the conditional confidence coefficient is approximately constant for the likelihood regions, though not for the Wilks-Daly regions.

The analysis is extended to the case that the residual sum of squares is itself used to estimate σ^2. In order to improve the linearity of the model the author shows how to find a non-linear transformation of the parameters.

(C. L. Mallows)

BEALE, E. M. L. (Report of a discussion on a paper by) **4.2** (4.4)

Confidence regions in non-linear estimation—*In English*

J. R. Statist. Soc. B (1960) **22**, 77-78 (2 references, 1 figure)

The contributors to the discussion were: Mallows, Tocher, Barnard, Good, Cox, Plackett, M. S. Bartlett, Paddle, Goodman and H. Stone. The author replied.

Mallows showed that, in the situation considered by the author, the approximate effect of small departures from normality could be evaluated. Departures from isotropy may also be important; in such a case some discussion of principal curvatures would be needed. The Wilks-Daly regions would be regarded as probability regions by a statistician who happened to assume exactly the right prior distribution.

Tocher questioned whether the ideas of confidence intervals were useful. In two or more dimensions, it would be desirable to think of a given region, or shape of region, and to be able to assign a probability to it. If one accepts that the likelihood function gives the natural shape, the problem becomes one of designing the experiment to achieve the desired shape.

Barnard remarked that the likelihood regions do convey the general course of the likelihood function; the associated confidence coefficient is approximately a fiducial probability if the parameters are completely unknown. With modern computors the whole likelihood function can be explored.

The confidence argument was discussed by Good in the context of the interplay between statisticians, sample customer, and action. He described a compromise of non-Bayesian and Bayesian interval estimation procedures, and investigated the utility of specifying a final probability density for a parameter. A confidence statement will be a reasonable guide to action if the loss is not too large.

Cox discussed regions for vector parameters in general and with reference to orthogonal comparisons in the analysis of variance. The specification of confidence coefficients for multiple statements offers conceptual difficulties, but these disappear if the statements are generated from the likelihood region.

Plackett pointed out that the least-squares estimate is not unique and M. S. Bartlett mentioned a particular problem in which a linearising transformation had been successful. Paddle described his research work into the likelihood function in the case of the Mitscherlich law and Goodman commented on the word " serendipity " and its implications for the author's paper. H. Stone described how the likelihood region could be used to generate a set of multiple confidence intervals.

(C. L. Mallows)

DEMING, W. E. & GLASSER, G. G. (New York University) **4.9** (8.7)
On the problem of matching lists by samples—*In English*
J. Amer. Statist. Ass. (1959) **54**, 403-415 (6 references, 2 tables)

The authors of this paper postulate the situation in which one has two or more lists of names and wishes to know how many names are common to some or all of the lists. Such information would be of importance to an advertising agency if, for example, they wished to decide whether to use one or both of two mailing lists. Earlier work by Goodman on this subject is both applied and extended [see *Ann. Math. Statist.* (1952) **23**, 632-634].

If samples of size m, n are taken from populations of size M, N, then the number D of names common to both populations is estimated by $\hat{D} = dMN/mn$, where d is the number common to both samples. The proportion p of names in the first population which occur in the second is estimated by $\hat{p} = dN/mn$. The distribution of d is derived, and its first four moments are given. The corresponding properties of $\hat{D}$ and $\hat{p}$ follow immediately, of course. Two limiting cases are considered:

(i) M, N, m, n all increase without limit in such a manner that D, m/M, n/N remain fixed.

(ii) M, N, m, n and D all increase without limit in such a manner that mnD/MN remains fixed. In the extension to L lists, D is taken to be the number common to all lists.

Optimum sample sizes are given in terms of cost of sampling and cost of matching samples. The problem of duplication of names within a single list is discussed. Finally, the authors consider the gain in efficiency by stratification: for example, by last initial or geographic location.

Several examples of estimation and testing hypotheses are presented.

(R. J. Buehler)

HAIGHT, F. A. (Inst. Transportation & Traffic Eng., Univ. California, Los Angeles) **4.3** (2.5)
The generalised Poisson distribution—*In English*
Ann. Inst. Statist. Math., Tokyo (1959) **11**, 101-105 (8 references, 2 tables)

The problem which the author treats in this paper is to give a simple expression for the mean m and to estimate the parameter λ of the generalised Poisson law

$$\Pr(N = n) = \sum_{i=1}^{k} e^{-\lambda}\lambda^{nk+i-1}/(nk+i-1)!$$

where k is some positive integer. Starting from the probability generating function of this distribution, the author gives the expression for the mean m in the form $m = \sum_{n=2}^{\infty} a_n\lambda^n/n!$ $(k>1)$ and the table for the coefficients a_n is given for $k = 2$ (1) 12 and $n = 1$ (1) 14.

In estimating λ from data, the method of moments is used. When $k = 2$, a simple approximate method is given: that is, λ is estimated for a given m from the approximate relation $\lambda \approx 2m+\frac{1}{2}$. For values of k greater than 2, the author gives the relation $\lambda(k) \approx km+\frac{1}{2}(k-1)$ for estimating λ from m, based on the assumption that for fixed m, λ is a linear function of k. The paper states that this linearity has been found in every numerical problem that the author has investigated and that the equations above have always proved accurate to three places. A numerical example is given which uses the data of Barton & David on the dates of consecration of Archbishops of Canterbury.

(M. Siotani)

HARTLEY, H. O. (Statist. Lab., Iowa State College, Ames) **4.3** (10.7)

The efficiency of internal regression for the fitting of the exponential regression—*In English*

Biometrika (1959) **46**, 293-295 (7 references, 1 table)

Finney [*Biometrika* (1958) **45**, 370-388: abstracted in this present journal, No. 48, **4.3**] declared himself unable to understand what advantage could be claimed for the use of the author's method of " internal regression " for the estimation of the base parameter ρ in an exponential regression, since it is considerably more laborious than other methods, and " takes no account of the pattern of errors ".

In this paper the author refutes this assessment. In his original paper [*Biometrika* (1948) **35**, 32-45] he did discuss the pattern of errors; two possible assumptions are (i) that the residuals from the basic exponential regression are independent and normally distributed with common variance, and (ii) that this is true of the residuals from the derived autoregression equation. He was concerned mainly with (i) and deliberately decided against using the estimator advocated by Finney which has been examined by Patterson [*Biometrika* (1958) **45**, 389-400: abstracted in this present journal No. 55, **4.3**] and is considered by Finney himself to be severely restricted in general use.

The asymptotic efficiency of the internal regression estimator is tabulated for various values of $-n\ln\rho$.

Finally, the question is raised of the robustness of the various estimators when the true error pattern departs from that assumed.

(C. L. Mallows)

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IRWIN, J. O. (London School of Hygiene) **4.3** (2.5)

On the estimation of the mean of a Poisson distribution from a sample with the zero class missing—*In English*

Biometrics (1959) **15**, 324-326 (4 references, 1 table)

The maximum likelihood estimator of λ for the case of a sample from a Poisson distribution with the zero class frequency missing is $\bar{x} = \lambda/(1-e^{-\lambda})$ where $\bar{x}$ is the sample arithmetic mean. An implicit solution (given $\bar{x}$) with a table for $\hat{\lambda}$ was given in a paper by Florence N. David & Johnson [*Biometrics* (1952) **8**, 275-285]. They also wrote that an explicit expression did not seem possible—a view that was supported by Finney & Varley [*Biometrics* (1955) **11**, 387-394] who gave a rapid method of solving the above equation by iterative and interpolatory procedures.

This note puts forward an explicit expression in the form of a Lagrange series:

$$\hat{\lambda} = \bar{x} - \sum_{r=1}^{\infty} \{r^{r-1}(\bar{x}e^{-\bar{x}})^2/r!\}$$

When $\bar{x} = 1$ the sum of the exponential series is also unity: therefore $\hat{\lambda}$ is zero. The author notes that the series may be awkward for $1 \leqq \bar{x} \leqq 2$ but should converge satisfactorily for higher values.

Attention is also drawn to the method proposed by McKendrick [*Proc. Edin. Math. Soc.* (1926) **44**, 1-34] and the general paper on maximum likelihood estimation from incomplete data by Hartley [*Biometrics* (1958) **14**, 174-194].

(W. R. Buckland)

KAHN, L. B. (Shell Development Co., Emeryville, California) **4.3** (2.5)

A statistical model for evaluating the reliability of safety systems for plants manufacturing hazardous products—*In English*

Technometrics (1959) **1**, 293-307 (11 references, 1 table, 9 figures)

This paper outlines a technique for determining the probability of failure of a plant manufacturing a hazardous product. For the failure of the plant to occur, the failure of the safety system must precede the failure of the plant's critical operating system in any " turnaround " or inspection period; where the failure of the operating system would precipitate a hazardous situation. The probability of plant failure is calculated on this conditional basis. The function for determining the probability of plant failure in a turnaround period, T, follows:

$$\Gamma_s(T, \gamma) = \int_0^{T-\gamma} \phi_c(t)\Phi_s(t)dt, \quad 0 \leqq \Gamma_s(T, \gamma) \leqq 1,$$

where $\phi_c(t)$ is the probability density function for operating system failure, $\Phi_s(t)$ is the probability distribution function for safety system failure, T is the turnaround or inspection period, and γ is the " grace-time ", or time length from the occurrence of the primary failure of the operating system to possible plant disaster. The subscript s denotes the number of components comprising the safety system.

Since the probability of plant failure must be related to the " life of the plant, L," then

$$\Gamma_s(n, L) = 1-[1-\Gamma_s(T, \gamma)]^n, \quad 0 < T \leqq L, \quad n = L/T.$$

Thus, for continuous inspection during the life of the plant,

$$T \to 0, \quad n \to \infty, \quad \Gamma_s(n, L) = 0.$$

Where no inspection is conducted during the life of the plant, $T = L$, and $\Gamma_s(1, L) =$ a calculated probability.

The author has illustrated the use of the functions $\Gamma_s(n, L)$ and $\Gamma_s(T, \gamma)$ for differently designed safety systems, showing the probable risk of disaster, where the components are given first in series and then in parallel. Under the assumption that the exponential theory of failure applies, probabilities are calculated, showing the effect of changing component reliability and turnaround periods, giving an insight into the problem of redundancy.

Examples of disastrous occurrences in industrial practice are cited, these point to the need and applicability of a scientific approach to safety system evaluation in these industrial areas.

(L. B. Kahn)

KAO, J. H. K. (Cornell University, Ithaca, New York) **4.3** (2.5)

A graphical estimation of mixed Weibull parameters in life-testing of electron tubes—*In English*

Technometrics (1959) **1**, 389-407 (8 references, 2 tables, 8 figures)

In broad terms electron tube failures can be classified into two types: catastrophic or sudden failures and wear-out or delayed failures. The catastrophic failures include glass failure, short circuits and open circuits: there are also indirect causes. For example, the current in a tube which has a slow electrical leak may rise excessively and cause a tube element (e.g. screen grid) to melt and short-circuit to other elements.

For most applications, since it is usually the poor-quality tubes that fail prematurely: the failure rate of the population decreases with operation on life. Furthermore, it is reasonable to assume that failures of these poor-quality tubes would actually start as soon as they are exposed to the risk of failure. Under these considerations a Weibull distribution, with location parameter equal to zero and shape parameter not greater than unity indicating a constant or decreasing failure rate, would be an appropriate statistical model.

The most important causes for wear-out failure are of an electrical or chemical nature. The wear-out failure is associated with the " wearing-out " of one or more of tube's elements. In the actual use of tubes, the sudden decrease of tube characteristics causes " wear-out " failure according to applications which include circuit requirement —such as a variable duty-cycle, environmental stress—such as elevated temperature, maintenance policy—such as marginal checking, legal regulations—such as those imposed on airborn applications, etc. For most applications, very few wear-out failures occur until some particular time period has elapsed after which the sudden decrease of characteristics causes the failure rate to increase with time. The length of this delay period depends upon application and in the case of static life-testing, upon the specific definition of the end-point of the life-test. In view of these considerations a Weibull distribution with some positive location parameter (delayed failure) and shape parameter larger than unity (increased failure rate) seems to be a reasonable model. Combining the above two failure causes, a mixture of two Weibull populations is postulated as the underlying distribution of the life-time of electron tubes.

A two-fold mixed Weibull cumulative distribution function is defined as:

$$F(x) = \begin{cases} 0 & \text{for } x < \gamma_1 \\ p - pg_1(x) & \text{for } \gamma_1 \leqq x \leqq \gamma_2 \\ 1 - pg_1(x) - (1-p)g_2(x) & \text{for } \gamma_2 \leqq x < \infty \end{cases}$$

where $g_i(x) = \exp -[(x-\gamma_i)^{\beta_i}]/\alpha_i$, $\gamma_1 < \gamma_2$ real, α_i and β_i are real positive and $0 \leqq p \leqq 1$.

Life qualities such as moments, reliable life, failure rate, life expectancy etc., are defined in statistical terms and a graphical method of estimation from failure data using the Weibull probability paper is given. Tube failure data from a small scale life test, covering a running time of more than 5000 hours on some 800 tubes, are used as illustrations. The construction of confidence band using the incomplete beta function table is also described and is appended with some proofs relevant to the paper.

(J. H. K. Kao)

MORANDA, P. B. (Autonetics, California) **4.1** (4.8)

Comparison of estimates of circular probable error—*In English*

J. Amer. Statist. Ass. (1959) **54**, 794-800 (1 reference, 3 tables)

This paper compares the variances of four different estimators of the circle of equal probability (CEP). The CEP (also called Circular Probable Error) measures the radius of the mean-centred circle which includes 50 per cent. of the bivariate probability mass: it is commonly used in describing the accuracy of a weapon. Under the assumption that x and y errors (or deviations from the aim point) are independently and normally distributed with mean zero (known) and common variance σ^2 (unknown), the CEP is given by $1{\cdot}1774\sigma$. [Equation (4) in the paper is in error. It should read CEP $= 1{\cdot}1774\sigma$].

The maximum likelihood estimate, adjusted to correct the bias, is denoted by $\widehat{\mathrm{CEP}}_1$. A further maximum likelihood estimate, obtained without assuming mean zero, but also corrected for bias, is denoted by $\widehat{\mathrm{CEP}}_2$. A third estimate, which is the sample median of observed radial errors, is denoted by $\widehat{\mathrm{CEP}}_3$. A fourth estimate, which is simply the arithmetic mean of observed radial errors is denoted by $\widehat{\mathrm{CEP}}_4$.

In table 798, the variances of the above four estimates are given corresponding to sample sizes 2(1)21. It is seen that $\widehat{\mathrm{CEP}}_1$ has the smallest variance, as expected, since Chapman & Robbins [*Ann. Math. Statist.* (1951) **22**, 581-586] have shown such an estimate to be " best unbiased ". $\widehat{\mathrm{CEP}}_2$ and $\widehat{\mathrm{CEP}}_4$ have a somewhat larger variance, while $\widehat{\mathrm{CEP}}_3$ has the largest variance of all the estimators considered.

Table 799a gives the efficiency of the 4 estimates, relative to that of the best estimate $\widehat{\mathrm{CEP}}_1$ and table 799b gives the corrective factors required to make the estimates unbiased. The sample sizes considered are 2(1)21.

(J. Gurland)

601

PATTERSON, H. D. & LIPTON, S. (Rothamsted Experimental Station, England) **4.3** (10.7)

An investigation of Hartley's method for fitting an exponential curve—*In English*

Biometrika (1959) **46**, 281-292 (8 references, 10 tables)

The bias and efficiency of Hartley's method of " internal regression " for estimating the base parameter ρ in an exponential regression are investigated. It is assumed that the n observations y_x are independently normally distributed with common variance.

A " linear estimator " is defined to be the ratio of a certain weighted average of the y's (omitting the first) to the same weighted average (omitting the last). In a " quadratic estimate " the weights are themselves linear functions of the y's. For each type, the weights can be chosen to give an estimate with minimum variance for one particular value of ρ. The minimum-variance linear and quadratic estimates are identified approximately using first-order expressions for the biases and variances. These expressions are less appropriate for use with the least-squares estimate.

It is shown that Hartley's method gives a quadratic estimate, belonging to the family having full efficiency as ρ approaches unity. Other " internal regression " methods are discussed.

For $n = 4, 5, 6, 7, 9, 12, 20$ and infinity, and for various values of ρ, the biases, standard errors, and efficiencies are tabulated for the least-squares estimate and for several members of the above family of quadratic estimates. The efficiencies only fall away from unity when n is large and ρ small. It is shown that, while Hartley's method generally leads to estimates having relatively high efficiencies and small biases, this is not true of similar methods which are sometimes used in place of the original Hartley method.

A note added in proof presents another method of estimation which it is believed will have uniformly high efficiency.

(C. L. Mallows)

602

RAO, J. N. K. (Iowa State University, Ames) **4.3** (3.6)

A note on mean square successive differences—*In English*

J. Amer. Statist. Ass. (1959) **54**, 801-806 (4 references, 3 tables)

This paper deals with the use of higher order mean-square successive differences for estimating dispersion when a strong trend is present in the mean value and the method of first differences does not adequately eliminate the trend. The first four moments of the mean-square successive second difference, δ_2^2, are derived in samples from an arbitrary population with constant mean (or with linear trend in the means). Because the efficiency of δ_2^2 relative to s^2 increases with β_2, the approximate distribution of δ_2^2 is discussed only for a leptokurtic population, specifically for the symmetrical two-tailed exponential population. For populations with varying mean, the first two moments of δ_2^2 are given and the effect of trend of the mean values on these moments is discussed.

Short tables of the efficiency of the mean-square successive second, third, and fourth differences, all relative to s^2 are given.

(J. N. K. Rao)

603

SARHAN, A. E. & GREENBERG, B. G. (University of North Carolina, Chapel Hill) **4.3** (2.6)

Estimation of location and scale parameters for the rectangular population from censored samples—*In English*

J. R. Statist. Soc. B (1959) **21**, 356-363 (4 references, 3 figures)

The authors consider the case of the so-called Type II censoring, that is to say when the censoring is done on the number of observations collected and not on some previously defined value of the measured variate (Type I). Consideration is given to the question of obtaining linear estimates of the location and scale parameters of a rectangular distribution from a sample of n items given that the smallest r_1 observations and the largest r_2 observations are censored.

If θ_2 is the range of the rectangular distribution and θ_1 its centre, it is shown that the estimators of θ_1 and θ_2 are the weighted functions of the difference and the sum respectively of the largest and the smallest recorded observations. The variances of these estimations are given and their efficiencies relative to the complete uncensored sample.

The standard error of the estimates of the start and the end of the distribution and the special case where the start is known and their efficiencies are also given.

Figure 1 gives the relative efficiency of estimating the mean of the rectangular distribution for a sample of size 10 when $r_1+r_2 = 1$ (1)...8. Figure 2 for similar data gives the relative efficiency of estimating the range, and Figure 3 for the starting point.

(Florence N. David)

604

SHENTON, L. R. (College of Science and Technology, Manchester) **4.1** (4.3)
Moment estimators and maximum likelihood—*In English*
Biometrika (1959) **46**, 502

The author notifies two corrections on page 415 of his paper already abstracted in this journal No. 58, **4.1.**

(W. R. Buckland)

TATE, R. F. & KLETT, G. W. (University of Washington, Seattle) **4.4** (11.1)
Optimal confidence intervals for the variance of a normal distribution—*In English*
J. Amer. Statist. Ass. (1959) **54**, 674-682 (7 references, 2 tables)

The authors consider several different types of confidence intervals for the variance of a normal population. Let $X_1, X_2, \ldots, X_N$ be a random sample from $N(\mu, \sigma^2)$, μ and σ^2 unknown. Denoting the confidence interval (I) and the confidence coefficient (ϵ), the authors restrict themselves to intervals of the form

$$I = \left[\frac{1}{b_n}\ \Sigma(X_i - \bar{X})^2,\ \frac{1}{a_n}\Sigma(X_i - \bar{X})^2\right]$$

where $n = N-1$.

The restrictions on a_n and b_n are discussed if I is to be (1) minimum length, (2) shortest unbiased (Neyman sense), (3) likelihood ratio, (4) " equal tails ". No claim is made about the originality of the conditions. It is pointed out that the shortest unbiased interval is the unique unbiased interval of the above type; hence all the other types are biased. The minimum length interval is in many instances shorter than the shortest unbiased interval. The " equal tails " interval is computationally easier but is in no sense optimal. There appears to be nothing to recommend the likelihood ratio interval: all the intervals are equivalent for large n.

Tables are presented which give (a_n, b_n) for both minimum length intervals and shortest unbiased intervals for $n = 2(1)29$ and $\epsilon = (0{\cdot}900, 0{\cdot}950, 0{\cdot}990, 0{\cdot}995, 0{\cdot}999)$. Two examples are worked showing the minimum length interval, the shortest unbiased interval and the " equal tails " interval.

A short discussion of the derivation of the tables is given and bounds of error presented.

(H. Larson)

WATTERSON, G. A. (Australian National University, Canberra) **4.8** (6.1)

Linear estimation of censored samples from multivariate normal populations—*In English*

Ann. Math. Statist. (1959) **30**, 814-824 (8 refs., 4 tables)

In this paper, the known methods of linear estimation are extended to various cases of censored samples from multivariate normal populations. Censorship effective on interior as well as extreme variates is allowed. The two estimators considered correspond to the minimum variance and the "alternative" estimators treated by Gupta, Sarhan and Greenberg for the univariate case. The alternative linear estimators referred to have coefficients which are obtained by assuming the variance matrix of the order statistics from a standard normal population to be the unit matrix. Such a set of coefficients α_i minimises $\Sigma\alpha_i^2$.

The method presented requires the sample to be ordered with respect to one variate, say x_1, and the remaining variables are associated with this arrangement. The author classifies censoring into three distinct types:

Type A: censoring effective on both the ordered variate and its associated variates,

Type B: censoring of associate variates only, and

Type C: censoring of the ordered variate only.

For the bivariate case considering the ordered variate x_1 and an associated variate x_2, say there are five parameters requiring estimation, namely μ_1, μ_2, $\sqrt{\sigma_{11}}$, $\sqrt{\sigma_{12}}$ and σ_{12}. Linear unbiased estimators of μ_1, μ_2, $\sqrt{\sigma_{11}}$, $\sigma_{12}/\sqrt{\sigma_{11}} = \rho_{12}\sqrt{\sigma_{22}}$, and σ_{12} (although the estimator of the latter is not strictly linear but is a non-linear function of the linear functions of the two samples and the covariances of ordered standard normal variables) are presented, and for Type C censoring a linear unbiased estimator of $\sqrt{\sigma_{22}}$ is also available. However, a linear estimator which has minimum variance has coefficients which are functions of ρ_{12}^2. The author therefore considers a set of estimators which are unbiased and of minimum variance when $\rho_{12}^2 = 1$.

These estimators are compared with the "alternative" estimators and, unless $|\rho_{12}|$ is very near unity, the alternative estimators are more efficient than the original, and in any case are never substantially less efficient. For a doubly-censored bivariate normal sample of type A or B of size ten the relative efficiencies of the two types of linear estimators of both μ_2 and $\rho_{12}\sqrt{\sigma_{22}}$ showing minimum and maximum values, and the correlation required for equal efficiency are tabulated. Also tabulated, for the same type sample and for $\rho_{12} = 0$, are the minimum efficiencies of both linear estimators of μ_2 relative to all linear estimators.

The multivariate case can be deduced by applying the theory to each pair of variates.

(J. W. Wilkinson)

607

WHITTLE, P. (Applied Maths. Lab., D.S.I.R., New Zealand) **4.5** (-.-)

Continuous generalisation of Tchebycheff's inequality—*In English*

Teor. Veroyat. Primen. (1958) **3**, 386-394 (4 references)

Given a random function $x(t)$ $(0 \leqq t \leqq 1)$ for which one knows only $\mathscr{E}[x(t)]$ and, $\mathscr{E}[x(s)x(t)]$ $(0 \leqq s, t \leqq 1)$, a bound is presented for the probability that $|x(t)|$ anywhere exceeds a given function $\alpha(t)$. The bound involves an arbitrary quadratic form. Let $S(x, y)$ be a symmetrical bi-linear functional $S(x, y) > 0$ for non-zero x; A and B are operators:

$$S(x, x) = \int_0^1 |Ax(t)|^2 dt, \quad Bx(t) = \int_0^1 B(t, s)x(t)ds$$

and have same properties, then

$$\Pr\{|x(t)|^2 \leqq B(t, t);\ 0 \leqq t \leqq 1\} \geqq 1 - \mathscr{E}[S(x, x)].$$

The case in which x is a function of several variables is also considered.

(B. V. Gnedenko)

WIJSMAN, R. A. (University of Illinois, Urbana) **4.2** (4.3)
On the theory of BAN estimates—*In English*
Ann. Math. Statist. (1959) **30,** 1265-1270 (1 reference)

The author notifies an error in the proof of Theorem 1 of his paper [*Ann. Math. Statist.* (1959) **30**, 185-191: abstracted in this journal No. 408, **4.2**]. As printed this theorem is not generally true and requires stronger assumptions—either on the distributions of the random vectors Z_n or on the estimator $\hat{\theta}$. A modification to Definition 2 is stated as Definition 3 and it is remarked that Theorem 2 is still true despite the stronger Definition 3. A proof of this theorem is given under the strong assumption of Definition 3.

(W. R. Buckland)

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610

AIVAZIAN, S. A. 5.7 (-.-)

A comparison of the optimal properties of the Neyman-Pearson and the Wald sequential probability ratio tests—*In Russian*

Teor. Veroyat. Primen. (1959) **4**, 86-93 (9 references)

Let H_θ be a hypothesis that the theoretical probability density is $f(x, \theta)$, where $\theta = (\theta_1, \ldots, \theta p)$ is an unknown parameter. The statistical problem of distinguishing between the two hypotheses H_0 and H_θ provided $\theta \to \theta_0 \doteq 0$ is considered.

The paper gives asymptotic formulae for the necessary number of observations when the Neyman-Pearson test is used and for the mean number of necessary observations for the Wald sequential test.

The limit of the ratio of these two numbers depends only on α and β (probabilities of the errors of first and second kind).

The variation of this limit with α and β is illustrated by a table at the end of the paper.

(S. Aivazian)

611

ARMITAGE, P. (London School of Hygiene) 5.0 (5.7

The comparison of survival curves—*In English*

J. R. Statist. Soc. A (1959) **122**, 279-292 (14 references, 2 tables, 1 figure)

In this paper the author examines the asymptotic relative efficiencies of four methods for comparing two distributions of the time of survival, or life. The comparison is made under the assumption that the underlying distributions are exponential and that the individual units, or sample members, enter the field of study at a uniform rate ($2n/T$), during the time interval (0, T): analysis of the accumulated results takes place at time T. The four methods compared are:

(i) maximum likelihood—a parametric, efficient method.

(ii) the sign method, in which individuals entering together are paired and their survival times correspond: a non-parametric test.

(iii) comparisons of the proportions of survivors at a given age (T) from entry—using only those entrants in the interval (0, $T-\tau$).

(iv) an actuarial method based upon the life-table: essentially distribution-free.

The asymptotic relative efficiencies are expressed as functions of λT, where λ is the death-rate per unit time. The author concludes that the sign method (ii) is particularly suitable for the sequential approach: the practical requirement (in connection with clinical trials for the treatment of chronic diseases) which gave rise to this investigation. The results, however, are valid over a wider area of application. Reference is made, in particular, to previous work by Littell [*Hum. Biol.* (1952) **24**, 87-116] who compared the relative efficiencies of (parametric) maximum likelihood and various actuarial methods.

A table is given of values of the efficiency index $\psi(\lambda T) = \chi^2(\lambda/\delta\lambda)^2 n^{-1}$ for various values of λT(0, 0·1, 0·2, 0·5, 0·8, 1·0, 2·0, 5·0, 10·0, 20·0 and ∞) and the different methods of analysis. In connection with the third method, values of ψ are given at four different levels of $\lambda\tau$ and for the optimum value of τ. For the fourth method the values of ψ are given in relation to the optimal value of τ using the product-limit method defined by Kaplan & Meier [*J. Amer. Statist. Ass.* (1958) **53**, 457-481] but actually in use from 1912 (Böhmer). The optimum values of τ, and the corresponding efficiencies, are shown in Table 2 of the paper. In both tables, figures are given for the percentage asymptotic relative efficiencies using the (parametric) maximum likelihood method as a standard. The basic values of ψ, the efficiency index as given in Table 1, are plotted in a graph.

An example is given together with some discussions of the results in relation to the applicability of methods to the sequential approach. It is emphasised that the effect of departures from the assumption of an exponential distribution is unknown.

(W. R. Buckland)

612

ARMITAGE, P. (Report of a discussion on a paper by) **5.0** (5.7)
The comparison of survival curves—*In English*
J. R. Statist. Soc. A (1959) **122**, 292-300

The discussion on this paper consisted of eight verbal contributions, an additional contribution in writing received after the meeting and the author's reply. Those who took part, in order of speaking, were: Cox, Boag, Bradford Hill, Spicer, Hajnal, Tanner, Winsten and Barnard. The written contribution was from Bartholomew.

After citing some examples of the comparison of survival curves in substantive fields other than the medical one, Cox considered the construction of sequential tests based upon maximum likelihood estimates and making use of the parametric formulation given in the second section of the paper. His argument involved four assumptions, viz.:

(i) use of asymptotic sufficiency instead of exact sufficiency,

(ii) use of asymptotic density,

(iii) use of estimates of nuisance parameters,

(iv) use of Wald's approximate formulae for limits,

and these were worthy of some investigation, particularly in relation to an earlier paper by the same author dealing with restricted sequential procedures [*Biometrika* (1957) **44**, 9-26].

Boag suggested that, while the exponential distribution was frequently applicable, consideration should be given to the lognormal distribution and Bradford Hill discussed some of the practical difficulties of conducting the kind of clinical trials which were involved. Spicer advocated the use of the life-table method when information upon the distribution of survival-time was lacking. He also pointed out that some explanation was needed of why the lognormal distribution was valid in those cases where it appeared adequately to describe the available data on a particular problem.

After making reference to the consequence of the condition for the sign-test, i.e., that one of the two survival curves is a fixed-power of the other, Hajnal referred to what would be the common-sense approach to the case where λT was large so that almost all of the sample units died, or failed, before the end of the period of observation. Having noted that an ordinary t-test of the difference between mean survival-time would lose some data—which could be allowed for by using an actuarial method—he draws attention to the fact that differences between the survival life of paired sample units would increase the amount of information and, hence, give a more efficient method.

Tanner referred to a non-medical situation where the units were put on test serially over a period of time and Winsten commented on the essential difference in the nature of " costs " between a trial in the medical world and, say, an industrial experiment. Barnard drew attention to the

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use of the Kolmogoroff statistic in this context. Where the third method of the paper deals with the comparison of survivor-rates at some fixed point during the whole period, the Kolmogoroff statistic compares the maximum difference between the survivor-rates.

In a written contribution, Bartholomew examined the situation where the actual life, or survival, time was not known of those units known to have failed or died. This gives a quantal response aspect to the analysis. Methods (ii) and (iii) of the paper [see: Armitage, P. (1959) **5.0**, abstract No. 612] were not applicable and work was proceeding on a non-parametric estimate analogous to the product limit of Kaplan & Meier [*J. Amer. Statist. Ass.* (1958) **53**, 457-481].

The author's detailed reply, amplified in writing, included *inter alia* an interesting comment on the use of the expectation of life rather than survival proportions.

(W. R. Buckland)

BARTHOLOMEW, D. J. (University College of North Staffordshire, England) **5.4** (3.1)
A test of homogeneity for ordered alternatives, II—*In English*
Biometrika (1959) **46**, 328-335 (7 references, 6 tables)

In a previous paper [*Biometrika* (1959) **46**, 36-48: abstracted in this present journal, No. 222, **5.4**] the author introduced modified χ^2 and F tests for use against ordered alternatives. Given k normal variates with mean values $m_1, m_2, \ldots, m_k$, the test is for the hypothesis H_0, that these mean values are equal, compared with the alternative hypothesis H_2 that $m_1 \geqq m_2 \geqq \ldots \geqq m_k$, where at least one of the inequality signs is strict.

When the variances of the k variates are known, the distribution of the test criterion, denoted by $\bar{\chi}^2$, depends on certain probabilities $\Pr\{l, k\}$. In this paper use is made of a recurrence relation for these probabilities—proved by Miles [*Biometrika* (1959) **46**, 317-327]—to obtain percentage points of $\bar{\chi}^2$ in the case of equal variances and a table is given for $k = 3, 4, \ldots 12$. The values of β_1 and β_2 are also found for the case of equal variances.

This paper also suggests a two-sided test when it is possible to rank the means, under the alternative hypothesis, without knowing whether the sequence is increasing or decreasing. Tables of the percentage points for this test are given for $k = 3, 4$, for various combinations of the values of the variances. For equal variances and larger values of k the table for the one-sided test may be used if the significance level is doubled.

In the last section an approximation due to Plackett [*Biometrika* (1954) **41**, 351-360] is used to obtain $\Pr\{5, 5\}$ which gives good agreement with certain exact values.

(N. W. Please)

BHAPKAR, V. P. (University of North Carolina, Chapel Hill) **5.8** (4.8)
A note on multiple independence under multivariate normal linear models—*In English*
Ann. Math. Statist. (1959) **30**, 1248-1251 (3 references)

Given a random sample of n observations from a p-variate normal population, it is proposed to test the hypothesis that $\sigma_{ij} = 0$, $i \neq j$, $i = 1, 2, \ldots, p$, by use of the step-down procedure referred to in a paper by J. Roy [*Ann. Math. Statist.* (1958) **29**, 1177-1187; abstracted in this journal No. 236, **5.8**]. It is assumed that the p variates can be ordered by the experimenter, so that successive tests can be made. These tests are all F-tests (F_i) with i and $n-r-i$ degrees of freedom (there are r parameters in the mean of the population to be estimated), $i = 1, 2, \ldots, p-1$. The successive tests proceed as follows:

(i) Reject $H_{01} : \sigma_{12} = 0$ if $F_1 > f_1$, where f_1 is the upper α_1 significance point for F_1.

(ii) If H_{01} is not rejected, reject $H_{02} : \sigma_{13} = \sigma_{23} = 0$ if $F_2 > f_2$.

(iii) Continue until one $F_i > f_i$ or none is rejected. The final significance probability is $\alpha = 1 - \prod_i (1 - \alpha_i)$.

Simultaneous confidence bounds associated with these tests are also given.

(R. L. Anderson)

CARVALHO, P. E. de OLIVEIRA (University of São Paulo) **5.6** (10.1)

On the distribution of the Kolmogoroff-Smirnoff D-statistic—*In English*

Ann. Math. Statist. (1959) **30**, 173-176 (8 references)

This paper presents what the author believes is a new way of deriving the exact distribution of the Kolmogoroff-Smirnoff D-statistic.

Consider a random walk on the half plane $(t>0, s)$, starting from the origin, such that at every point (t, s) there are two possible steps to take, either to $(t+1, s+1)$ or to $(t+1, s-1)$ each with equal probability. For some positive integer n, consider the paths from the origin to the point $(2n, 0)$. Using the principle of reflection, it is shown that the number of paths which have a point on at least one of the two lines $s = \pm k$, where k is a non-negative integer $\leq n$, equals

$$2 \sum_{i=1}^{[n/k]} \binom{2n}{n+ik} (-1)^{i+1}.$$

Let $X = (x_1 < \ldots < x_n)$, $Y = (y_1 < \ldots < y_n)$ be two independent samples of ordered independent observations having the same continuous cumulative distribution function. Let $x_i \neq y_j$ for all i, j. Let Z be obtained by combining X and Y and arranging in the order of increasing magnitude. Let $S_n(x)$ be the number of observed values x_i which are less than or equal to x and $S'_n(x)$ the number of observed y_j's less than or equal to x. Then we define

$$D = \max_x \mid S_n(x) - S'_n(x) \mid.$$

Employing the fact, recognised by Gnedenko & Korolyuk, that a one-to-one correspondence exists between the set of all Z and the set of all paths in the random walk from $(0, 0)$ to $(2n, 0)$ which the author discussed, he then obtains $\Pr(D<k)$.

(M. V. Menon)

COLOMBO, B. (Faculty of Economics, University of Venice) **5.7** (5.0)

Notes on sequential methods—*In Italian*

Mem. Accad. Pat. Classe sci. mat. nat. (1959) **71**, 113-140 (23 references)

The author is concerned with demonstrating the flexibility of sequential procedures, and their adaptability to some current needs arising in scientific research. With this aim, he considers a number of classical problems in the testing of statistical hypotheses and shows how they can be treated sequentially, underlining the various advantages of such a course.

The points specifically considered refer to the testing of hypotheses about parameters of the univariate and bivariate normal distributions. Along the lines of Barnard, Cox and Rushton, explicit formulae are given for testing hypotheses concerning the variances of two normal populations when the samples can be of unequal numbers. An analogous result is presented in the case of the averages when variances are equal but unknown. A particular case of this is shown to be a sequential counterpart of the classical problem of the rejection of outlying observations. Applications are made to data from some research work in haematology.

Further sequential tests suggested in this paper refer to correlation and regression coefficients of two bivariate normal distributions: the results presented, valid only under heavy limitations, seem to give only partial solutions to the problem. Also unsatisfactory from a practical point of view, though formally correct, is the last proposed test giving a sequential treatment to classical tests like Hotelling's T^2 for multivariate problems concerning the means. According to the author, much work has still to be done, as also in the field of the sequential design of experiments, in order to give useful answers to many types of problem which arise in scientific work.

(G. Panizzon)

DEMPSTER, A. P. (Harvard University) **5.8** (5.3)

A significance test for the separation of two highly multivariate small samples —*In English*

Biometrics (1960) **16**, 41-50 (6 references)

This paper gives an explanatory derivation with formulas and illustrations of a statistical technique which has been mathematically discussed by the author [*Ann. Math. Statist.* (1958) **29**, 995; abstracted in this journal No. 226, **5.8**]. The technique provides an alternative to Hotelling's T^2 test of equality of mean vectors in two multivariate groups. It is particularly applicable in situations where the T^2 test is either invalid or is computationally difficult to apply. That is, when the sample size is too small ($k > n_1+n_2-2$, where k is the number of measurements and n_1, n_2 are the group sample sizes) or when the number of measurements is so large as to make the required matrix inversion impractical. In avoiding these difficulties, however, it is necessary to give up the desirable property of T^2 whereby the same T^2 results from any k linear combinations of the k variables used in place of the original k variables.

The test requires the assumptions of multivariate normality and homogeneity of dispersion matrices. The derivation of the test starts with an orthogonal transformation of the original data into a new set of n_1+n_2 vector observations such that the first is associated with the grand mean, the second with the hypothesis of equal mean vectors for the two groups and the remaining n_1+n_2-2 vectors with the residual. The proposed test statistic is the ratio of the length of the vector associated with the hypothesis to the average length of the vectors associated with the residual. The author proposes as an approximate distribution for this statistic, the F distribution with r and $(n_1+n_2-2)r$ degrees of freedom, where r, $(r<k)$, is a quantity to be estimated. Two methods for estimating r are given.

The test based on T^2 is unique under linear transformation of the original variables. The proposed test, however, does not possess this property and, therefore, the choice of variables becomes important. A discussion of this point is given by the author.

An example is presented using $k = 62$ biochemical measurements on $n = 12$ individuals and making use of the IBM 650. As an alternative the author suggests a randomisation test which is a generalisation of the Pitman randomisation test for the univariate two-sample t-test. This alternative test is applied to the same data as in the example and extremely close agreement with the results of the proposed test is obtained. On the basis of this, the author states that this "provides encouragement that the non-exact test based on normal distribution theory provides robust theory for attaching significance levels." The test proposed by Chung & Fraser [*J. Amer. Statist. Ass.* (1958) **53**, 729] is also applied to the same data but yields slightly lower significance levels.

(H. O. Posten)

619

GNANADESIKAN, R. (Bell Telephone Labs., Murray Hill, N.J.) **5.4** (5.8)

Equality of more than two variances and of more than two dispersion matrices against certain alternatives—*In English*

Ann. Math. Statist. (1960) **31**, 227-228 (2 references)

The author has published a brief note of correction to, and comment on, his paper already abstracted in this journal, No. 228, **5.4**.

He states that the region of acceptance of a test for the null hypothesis $H_0 : \Sigma_1 = \Sigma_2 = \ldots = \Sigma_k = \Sigma_0$ as stated in section four is wrong. It is the central result

$$\lambda_{j1} \frac{c_{\min}(S_j^*)}{c_{\max}(S_0^*)} \leqq \frac{c_{\max}(S_j^*)}{c_{\min}(S_0^*)} \leqq \lambda_{j2} \quad \text{for } (j = 1, 2 \ldots k)$$

which is the exact probability statement $\{\Pr(1-\alpha)\}$ and from which can be obtained the (implied) acceptance region

$$\frac{c_{\max}(S_j)}{c_{\min}(S_0)} \geqq \lambda_{j1} \quad \text{and} \quad \frac{c_{\min}(S_j)}{c_{\max}(S_0)} \leqq \lambda_{j2}$$

where

$$\lambda_{j1} < \lambda_{j2} \quad \text{and} \quad \frac{c_{\min}(S_j)}{c_{\max}(S_0)} \leqq \frac{c_{\max}(S_j)}{c_{\min}(S_0)}$$

The author also comments that a question raised by T. W. Anderson relating to test case with a different region of acceptance was, in fact, the starting point of his, the author's, investigations. It was, however, temporarily abandoned but is now being more fully investigated.

(W. R. Buckland)

620

HUDIMOTO, H. (Inst. Statist. Math., Tokyo) **5.3** (5.6)

On a two-sample non-parametric test in the case that ties are present—*In English*

Ann. Inst. Statist. Math., Tokyo (1959) **11,** 113-120 (4 references)

The author treats a two-sample non-parametric test in the case where information obtained from the sample is first A or non-A and then the condition of A, when A has occurred. Non-A is denoted by zero and the condition of A by a positive real number. For instance, the populations concerned are human populations and A represents that there are some in the sample who are subscribing to a newspaper, and the condition of A the proportion of newspaper readers in the sample.

The two distribution functions concerned $F(t)$ and $G(t)$ which are defined for $t \geqq 0$ are denoted by, say, f_0 and g_0 for $t = 0$, and $f_0+(1-f_0)F^*(t)$ and $g_0+(1-g_0)G^*(t)$ for $t>0$, respectively, where $F^*(t)$ and $G^*(t)$ are also distribution functions. The distribution functions $F^*(t)$, $G^*(t)$ are assumed to be continuous, but the observations obtained are grouped into a finite number of common intervals for the sake of conveniences of numerical treatment. Therefore, ties are present.

A statistic $\hat{h}(F, G)/(1-\hat{p})^2 = \sum_i[\hat{F}_i\hat{g}_i-\hat{G}_i\hat{f}_i]/(1-\hat{p})^2$ is adopted for the two sample problem where the alternative consists of at least one of the following situations: $(f_0<g_0)$ and/or $(F^*<G^*)$. The symbol ^ means that the quantity concerned is empirical, and $\hat{p} = \frac{1}{2}(\hat{f}_0+\hat{g}_0)$ in the case of equal sample sizes. The paper has considered only the case of equal sample sizes, but the similar procedure can also be carried out in the case of unequal sample sizes.

The test procedure is as follows. Determine $\epsilon>0$ and $\eta>0$ by means of some probablistic inequality [(*) of the paper] for a given significance level. Then, if

$$\{|\hat{h}(F, G)|-\epsilon/(1-\hat{p})^2\}>\eta,$$

reject the hypothesis $F(t) = G(t)$.

In the appendix, a necessary relation for evaluation of the inequality used for determining the above ϵ, η is mentioned using an analogous procedure to that given by Birnbaum & Tingey, " One-sided confidence contours for probability distribution functions " [*Ann. Math. Statist,* (1951) **22**, 592-596].

(H. Hudimoto)

621

ISHII, G. (Atomic Bomb Casualty Commission, Hiroshima) **5.6** (5.1)

On a non-parametric test in life test—*In English*

Bull. Math. Statist. (1959) **8**, 73-79 (9 references)

In this paper, for testing the hypothesis that the population distribution function is a continuous $F(x)$, against the alternative that the distribution is stochastically larger than $F(x)$, the following non-parametric procedure, based on a censored sample, is proposed.

Let M be the size of sample, and m and n integers such that $1<n<m$. The author takes as " stop time " (truncation point) the n/m-quantile $T(F(T) = n/m)$ of the distribution, and considers $(-\infty, T)$ as the sample space. Then, he divides the sample space into n intervals by i/m-quantiles $(i = 0, 1, \ldots, n)$ of the distribution, and counts the number v of intervals which contain no sample point. The null hypothesis is rejected if v is larger than a criterion.

Theorem 1 states the asymptotic normality of the distribution of v/n when M, m and $n \to \infty$ with constant ratios.

Theorem 2 states the unbiasedness of the test for such alternative distributions $F_1(x)$ as satisfies $F_1(T)<n/m$.

(M. Sibuya)

622

JAMES, G. S. (University of Leeds) **5.3** (2.7)

The Berens-Fisher distribution and weighted means—*In English*

J. R. Statist. Soc. B (1959) **21**, 73-90 (25 references, 3 tables)

The two-sample problem has been the subject of much controversy during the past few years. This is a contribution to this controversy. The author examines critical statements made by Fisher in his book [*Statistical Methods and Scientific Inference* (1956). Edinburgh: Oliver & Boyd]. The Berens-Fisher and Welch solutions of the problem are compared and, according to the criteria set up by the author, the former test is shown to be lacking in sensitivity. The author compares his own test for a weighted mean with that put forward by Yates and shows that Yates' test rejects the true value of the mean erroneously more often than is indicated by the true significance level.

A new exact test for weighted means is proposed. In this test, information on the variances supplied by the differences between the means is taken into account. The optimum properties of this new test are not discussed. The percentage points of the distribution of the criterion may be found from the Cornish-Fisher inversion of Edgeworth's series. The author takes this series for the case where the distribution is symmetrical to a greater number of terms than has previously been given.

(Florence N. David)

623

JOHNSON, N. L. (University College, London) **5.7** (4.7)

A proof of Wald's theorem on cumulative sums—*In English*

Ann. Math. Statist. (1959) **30**, 1245-1247 (5 references)

A simple proof is given that the expected value of the cumulative sum of a series of random variables, with a random number of terms (n) based on sequential analysis procedures, is the product of the mean of the random variables and the expected value of n. Given independent random variable $\{z_i\}$ with $\mathscr{E}(z_i) = \mu$, where $\mathscr{E}\,|\,z_i\,|$ is bounded. Define $Z_n = \sum_{i=1}^{n} z_i$, where n is a random variable, such that the event $\{n \geq i\}$ depends only on previous z's, that is to say on $z_1, z_2, \ldots, z_{i-1}$. Then $\mathscr{E}(Z_n) = \mu\mathscr{E}(n)$. The method used in the proof is also employed to evaluate the variance of n, assuming

$$\mathrm{Var}(z_i) = \sigma^2,\ \mathrm{Var}(n) = [\sigma^2\mathscr{E}(n) - \mathrm{Var}(Z_n)]/\mu^2.$$

(R. L. Anderson)

KEMPERMAN, J. H. B. (Purdue University, Indiana) **5.6** (10.1)

Asymptotic expansions for the Smirnoff test and for range of cumulative sums—*In English*

Ann. Math. Statist. (1959) **30**, 448-462 (6 references)

The Smirnoff test-statistic for distinguishing between two distribution functions on the basis of two equal-size independent samples $X_1, X_2, \ldots, X_n$; $Y_1, Y_2, \ldots, Y_n$ is $D_{nn} = \max_x |F_n(x) - G_n(x)|$, where F_n and G_n are the empirical distribution functions of the X and Y samples, respectively. It is known that $n[F_n(Z_i) - G_n(Z_i)]$, $(i = 1, 2, \ldots, 2n)$ is a Markoff chain with increments equal to ± 1; Z_i is the ith smallest observation among the combined X and Y samples. This Markoff chain may then be interpreted as a random walk by means of the reflection principle, and hence the Smirnoff test-statistic has a distribution related to that of the maximum of the first $2n$ values of this random walk.

In general, let $\{W_n : n \geqq 0\}$ denote a random walk on the integers with

$$\Pr[W_{n+1} - W_n = 1 \mid W_n] = \tfrac{1}{2} = \Pr[W_{n+1} - W_n = -1 \mid W_n].$$

Let

$$p_n(i, j, c) = \Pr[W_n = j, 0 < W_m < c, m = 1, 2, \ldots, n \mid W_0 = i],$$

where i, j, c, n are positive integers. With this notation, it has been shown by Gnedenko & Korolyuk [*Dokl. Akad. Nauk SSSR* (1951) **80**, 525-528] that

$$\Pr[D_{nn} < b/n] = \left\{ 2^{2n} \Big/ \binom{2n}{n} \right\} \{p_{2n}(b, b, 2b)\}.$$

In the present paper, the author develops an asymptotic expansion for $p_n(i, j, c)$ which yields for each integer m an approximation to it with absolute error less than Cn^{-m} where C does not depend on i, j, c, n. The author uses the known exact expression for $p_n(i, j, c)$, a short derivation of which is also included. The approximation obtained is of the form

$$(2/\pi)^{\frac{1}{2}} \sum_{k=0}^{m-1} n^{-k-\frac{1}{2}} \sum_{h=0}^{k} (-1)^h A_{kh} g_{k+h},$$

where A_{kh} is defined in terms of the first k Bernoulli numbers and where g_r is an infinite series whose summands are essentially values of the Hermitian polynomial of degree $2r$. This result is applied to the distribution of D_{nn}, to obtain approximations for it which extend earlier work of Gnedenko [*Dokl. Akad. Nauk SSSR* (1952) **82**, 661-663]. The author also uses his results to obtain both exact and asymptotic expressions for the distribution function of the range of the random walks first n values $W_1, W_2, \ldots, W_n$.

(R. Pyke)

KHATRI, C. G. (University of Baroda, India) **5.8** (4.8)

On the mutual independence of certain statistics—*In English*

Ann. Math. Statist. (1959) **30**, 1258-1262 (8 references)

In this note the author proves two theorems concerning the Wishart and the multivariate normal distributions: there are four corollaries to the first theorem and one corollary to the second. These results are stated to yield the mutual independence of certain matrices, characteristic roots, Hotelling's T^2 or Mahalanobis' D^2 statistics, and C. R. Rao's R statistic. They can be used in statistical inference and especially in simultaneous tests and confidence interval estimation.

(R. L. Anderson)

LINHART, H. (South African Council for Scientific & Industrial Res., Johannesburg) **5.8** (6.1)

A criterion for selecting variables in a regression analysis—*In English*

Psychometrika (1960) **25**, 45-58 (10 references, 4 tables)

This paper represents an extension of an earlier paper by this author introducing a criterion for selecting variables in a regression analysis. Procedurally the method is similar to the test selection method developed by Wherry. Linhart's proposal, however, provides for a test of the significance of the improvement in prediction from the addition of additional predictor variables.

The basis of the criterion is a measure for the precision of prediction. If one wants to predict x_0 by means of x_1, x_2, ..., x_k, one takes at first a regression sample of size n from the $k+1$ dimensional distribution of x_0, x_1, ..., x_k. This sample is then used to estimate the regression coefficients. A number of predictor sets are sampled randomly and the corresponding values of x_0 are predicted. For a fixed-regression sample and a fixed-predictor set, the length of a confidence interval for x_0, l, may be calculated. The expected value of l, $\mathscr{E}(l)$, where the expectation must be taken over all possible regression samples of size n and over all possible predictor sets, may be taken as a measure for the precision of prediction within the given population if regression samples of size n are used. It can be shown that $\mathscr{E}(l)$ may be smaller if only certain $k-r$ of the available k predictor variates are used, as compared with $\mathscr{E}(l)$ for all k predictor variates. Linhart points out, as have Wherry and others, that the precision of prediction may therefore deteriorate if more variables are used. An unbiased estimate of $\mathscr{E}(l)$ is given by:

$$\mathscr{E}(l) = [e_{kn}l_{00}(1-R^2)]^{\frac{1}{2}}\alpha_{kn},$$

where

$$e_{kn} = (n+1)/(n-k-1)[t_{.05}{}^{(n-k-1)}]^2;$$

$$\alpha_{kn} = 2\Gamma(\tfrac{1}{2}n)\Gamma[\tfrac{1}{2}(n-k-1)]/\Gamma[\tfrac{1}{2}(n-1)]\Gamma[\tfrac{1}{2}(n-k)];$$

$$nl_{00} = \sum_{r=1}^{n} (x_{0r}-\bar{x}_0)^2$$

are constants. For the test of the hypothesis $\mathscr{E}(l) \geqq \mathscr{E}_{[r]}(l)$ the statistic

$$\{R^2-R^2_{[r]}/1-R^2_{[r]}\}^{\frac{1}{2}}$$

may be used, which is distributed like a multiple correlation coefficient obtained from a sample of size $N = n-k+r$ out of a population with r independent variables having a multiple correlation

$$\{P^2-P^2_{[r]}/1-P^2_{[r]}\}^{\frac{1}{2}}.$$

P is the multiple correlation of x_0 on x_1, ..., x_k. The index $[r]$ always relates the symbol to which it is annexed to the case where r of the k variables are omitted. The author provides graphs of lower confidence limits for multiple P for 2 cases. An example of the application of this method is given.

(R. E. Stoltz)

627

PFANZAGL, J. (Universität Wien) **5.0** (5.4)

Tests and confidence intervals for exponential distributions and their application to some discrete distributions—*In German*

Metrika (1960) **3**, 1-25 (17 references)

The author derives significance tests, confidence intervals and point estimates for a class of distribution functions which are given by densities of the form $p(x, \theta) = C(\theta) \exp(\theta, x)$ with respect to a σ-finite measure μ defined on an abstract space. The exponential class is very important, as its is possible to transform most distributions which have a sufficient statistic x (with respect to θ) to the exponential form.

For this class x is a most powerful one-sided test statistic against $\theta > \theta_0$. The test statistic is in every case simultaneously used for the construction of confidence intervals and the bound of the one-sided confidence interval for θ with $\alpha = 0.5$ is used as the median unbiased estimate for θ. For construction of two-sided tests, the author used his test and classification method as given in a previous paper [*Metrika* (1959) **2**, 11-45: abstracted in this present journal, No. 235, **5.2**]. It is assumed that the distributions with the parameters $\theta_i+\Delta$ and $\theta_i-\Delta$ have the same *a priori* probability. When Δ is small, the locally optimum test statistic is given by $|x-\mathscr{E}_{\theta_0}(X)|$. In a similar way the two sample and the k sample problems are treated.

For the k sample problem, for example, with r different replications of an experiment, there are given k by r distributions with parameters θ_{ij} which are dependent upon the other parameters in the following way:

$$\theta_{ij} = \theta_j+\Delta_i(\delta) \text{ with } \Delta_i(0) = 0 (i = 1, ..., k; j = 1, ..., r).$$

Under the null hypothesis $\delta = 0$ and under the m different alternative hypotheses (with equal *a priori* probabilities) the functions $\Delta_{il}(\delta)$ $(l = 1, ..., m)$ are stated. The locally optimum test statistic is given by

$$\underset{l}{\mathrm{Max}}[\sum_i \Delta'_{il}(0) \sum_j \{x_{ij}-\mathscr{E}_0(X_{ij}/x_{.j})\}].$$

Some lemmas concerning the conditional expectation and the conditional variance are proved for the case that the distribution is reproductive with respect to a second parameter λ, and X is complete and sufficient for θ.

In the last section applications of the theory are given to the binomial, Poisson, negative binomial and Pascal distribution which are exponential and reproductive. For the difference of two Poisson distributions the method yields the test of Przyborowsky & Wilenski [*Biometrika* (1939) **31**, 313-323] and for the difference of two binomial distributions the two by two contingency table test of Fisher. The confidence interval for the difference is given in terms of $p_1/1-p_1 : p_2/1-p_2$. For the combination of two by two contingency tables the method yields a statistic which is used by Cochran [*Biometrics* (1954) **10**, 4147-451]. For the test against trend $(\Delta_i = i)$ in the binomial case the author gets a statistic which is, with the exception of a factor, identical with a special case of a statistic used by Yates for a two by k table [*Biometrika* (1955) **42**, 404-411].

(E. Walter)

628

PRATT, J. W. (Harvard University) **5.2** (2.6)

Admissible one-sided tests for the mean of a rectangular distribution—*In English*

Ann. Math. Statist. (1958) **29**, 1268-1271 (2 references)

The author proves the following theorem without resort to any general results from decision theory:

Sample of $n(>1)$ observations from a uniform distribution with mean θ and range R (known).

Test of hypothesis $H_0 : \theta \leqq \theta_0$ against $H_1 : \theta > \theta_0$.

An essentially complete class of admissible tests are those of the following type:

Let u = minimum observation

v = maximum observation

$g(u) = \theta_0 + \frac{1}{2}R$ for $u < \theta_0 - \frac{1}{2}R$.

Accept H_0 if, and only if, $v < g(u)$.

The two-sided problem has already been treated by A. Birnbaum [*Ann. Math. Statist.* (1954) **25**, 157-161].

(W. R. Buckland)

SMID, L. R. (Alge. Friesche Levens., Mij., Leuwarden and Univ. of Groningen) **5.6** (5.2)

Wilcoxon's test for symmetry when ties are present—*In Dutch*

Statist. Neerlandica (1959) **13**, 463-464 (4 references)

This communication is intended as a supplement to a paper by Benard, Constance van Eeden & Rümke [*Statist. Neerlandica* (1957), **11**, 231-235].

If n of the observations have values differing from zero, they are ranked according to their absolute values. Then the observations are grouped in ties of sizes $t_1, t_2, \ldots, t_k$ $(\sum_1^k t_i = n)$ successively. The observations of the jth tie are given the ranking number

$$r_j = \sum_{i=1}^{j-1} t_i + \tfrac{1}{2}(1+t_j).$$

If a_i of the observations of this tie have positive values and b_i negative values, then Wilcoxon's test statistic is defined by

$$T = \sum_1^k (a_i - b_i) r_i.$$

The author suggests that in practical cases the distribution should be calculated by making use of the generating function

$$V = 2^{-n} z^{-\frac{1}{2}n(n+1)} (1+z^{2r_1})^{t_1} (1+z^{2r_2})^{t_2} \ldots (1+z^{2r_k})^{t_k}.$$

The generating function has been derived by Constance van Eeden & Benard [*Proc. Kon. Ned. Acad. Wetensch.* A (1957) **60**, 384]. The method is applied to a numerical example where others have used a recurrence formual instead of a generating function.

(L. R. Smid)

WALSH, J. E. (System Development Corp., Santa Monica, California) **5.6** (9.1)
Exact nonparametric tests for randomised blocks—*In English*
Ann. Math. Statist. (1959) **30**, 1034-1040 (11 references)

In this paper a class of non-parametric procedures for testing the statistical identity of treatments in randomised block experiments is suggested and discussed. The suggested procedures are based on experimental within-block randomisations, and they may be chosen so as to have special power against particular alternatives. The blocks are assumed to be statistically independent but no assumption is made concerning dependence within the blocks. The observational data can be of any quantitative type.

Within a block, the assignment of the treatments investigated in that block can be of either a balanced or an unbalanced nature. For a given design, some blocks might be balanced and others unbalanced. The within-block assignments of treatments to locations are determined by a set of independent randomisation processes as follows: the treatments of each block are partitioned into disjoint classes; to each class there is assigned a set of eligible locations within the block; the assignments of treatments within a class to their eligible locations. For some classes, those of type A, the assignments are strictly random (all assignments equally likely), although possibly dependent from class to class but independent from block to block. A block always contains at least one class of type A and each of these contains at least two treatments. For the remaining classes, of type B the assignment to location may be random or fixed. The partitioning scheme, which may vary from block to block, is selected on the basis of the null-hypothesis and the alternative hypotheses being investigated.

In order to perform the test, a statistic is specified for each block. This statistic depends on all treatments for this block but not on those for any of the other blocks. These statistics are chosen so that they have symmetrical distributions about zero when the null-hypothesis is true. They are also chosen so that the test is sensitive to the alternative hypotheses that are emphasised. The forms of the statistics can vary from block to block but the block statistics are independent and have symmetrical distributions about zero under the null-hypothesis. Hence, the null-hypothesis can be tested by use of an appropriate non-parametric test of symmetry about zero. References to a wide variety of such tests are given. The author states that no generally applicable rules for choosing the block statistics can be stated but, in many cases, a reasonable selection can be made on an intuitive basis. The permissible forms for the block statistics, verification of their properties under the null-hypothesis, and a statement of how the test makes use of these statistics are presented. The author illustrates the method by two examples.

(R. Wormleighton)

631

WALTER, E. (Max-Planck-Institut, Göttingen) **5.1** (5.6)
Some properties of symmetry-tests—*In German*
Math. Ann. (1959) **137**, 433-453 (13 references)

The author introduces the concept of "efficiency of a class of tests" with respect to an alternative ω_1. This concept is to be understood as meaning that, for each $\delta \in \omega_1$ there exists a test in the class which is unbiased with respect to δ. After giving some general lemmas, the author considers tests for the hypothesis of symmetry with respect to zero. The sample space is the n-dimensional Euclidean space. Three different parameter-spaces are considered: Ω^u—the space of product measures, Ω^*—the space of measures invariant under all permutations of $(x_1, \ldots x_n)$, and $\Omega' = \Omega^u \cap \Omega^*$; that is to say, the space of product measures with identical components. Let ω be the class of measures which are symmetric with respect to zero; that is, invariant under any change of sign of the components of the point $(x_1, \ldots x_n)$. For each of the parameter spaces Ω defined above, the hypothesis $\delta \in \omega \cap \Omega$ is tested against the alternative $\delta \in \Omega - \omega$. The following theorems are proved:

(i) the class of tests of Neyman structure (i.e. conditional tests for given $[|x_1|, \ldots |x_n|]$) is complete for testing the hypothesis of symmetry against $\Omega^u - \omega$, and minimal complete against $\Omega^* - \omega$. Furthermore, this class is efficient with respect to $\Omega^u - \omega$.

(ii) further results are given for rank-tests (which are a special case of tests of Neyman structure): the class of rank tests is not efficient for testing the hypothesis of symmetry against $\Omega^u - \omega$, but it is efficient against $\Omega' - \omega$ for $n \geqq 3$. There exists no uniformly unbiased rank-test against $\Omega^* - \omega$.

(iii) a special class of rank-tests is considered, which was introduced by the author in an earlier paper using the test function $\sum_{j=1}^{r} g_j \,|\sum_{m_{j-1}+1}^{m_j-1} \delta_i\,|^{\mu}$, where the δ_i are the signs of $(x_1, \ldots x_n)$, ordered according to their absolute magnitude: $\mu \geqq 1$ and $g_j \geqq 0$ $(\Sigma g_j > 0)$ are arbitrary constants. In other words: the sample is ordered according to the absolute magnitudes of the observations and subdivided into r groups $(r \leqq n/2)$. For each group the excess of positive or negative numbers is raised to the power μ and multiplied by g_j. Finally, the scores of all groups are added. This test is shown to be efficient against $\Omega' - \omega$.

In proving the theorems mentioned above, the following lemma plays an important role: if for each $\delta \in \omega_1$ there exists a bounded function $g_\delta(x)$, such that

$$\mathscr{E}_\delta g_\delta(x) \begin{cases} = 0 \text{ for } \delta \in \omega \\ > 0 \text{ for } \delta \in \omega_1, \end{cases}$$

then any class of tests containing for each α, and each $\delta \in \omega_1$ the test function $\alpha + g_\delta(x)t$ is efficient against ω_1: t is a constant chosen such that $0 \leqq \alpha + g_\delta(x)t \leqq 1$.

(J. Pfanzagl)

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WEBER, E. (Humboldt-Universität, Berlin) **5.7** (5.2)

An introduction to sequential analysis and its application—*In German*

Wissensch. Zeit. Humboldt Universität (1957/58) **7**, 559-581 (5 references, 5 tables, 16 figures)

The paper starts with a brief introduction to the basic concepts of probability theory and of the testing of hypotheses. The subsequent chapters contain an elementary exposition of the theory of sequential tests of a single alternative against a simple hypothesis and its application for testing the parameter of a binomial distribution. For this case, the author discusses how the sequential test can be carried out numerically as well as graphically. Two numerical examples (one concerning an industrial, the other a biological application) are given. Finally, several methods for approximate and exact computation of the operating characteristic curve and average sample number curve are discussed and demonstrated for the numerical examples treated above.

A second paper is announced by the author for the succeeding volume of the same journal.

(J. Pfanzagl)

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WIJSMAN, R. A. (University of California, Berkeley) **5.1** (2.1)

Incomplete sufficient statistics and similar tests—*In English*

Ann. Math. Statist. (1958) **29**, 1028-1045 (17 references)

In this paper a method called the " D " (differential operator) is given for constructing a class of similar tests for a family of probability density functions called *regular exponential* in the case where the minimal sufficient statistic is boundedly incomplete. Further, the method provides a proof of the criterion for bounded incompleteness. In this sense the term " minimal sufficient " introduced by Lehmann & Scheffé [*Sankyā* (1950) **10**, 305-340 and (1955) **15**, 219-236] is equivalent to " necessary and sufficient " used by Bahadur [*Ann. Math. Statist.* (1954) **25**, 432-462]. The regular exponential densities are defined:

$$\mathrm{Pr}_\theta(t) = c(\theta)\exp[-\sum_{i+1}^{m} s_i(\theta)t_i]h(t).$$

The third section of the paper outlines the " D " method and in the fourth section are given examples relating to the Behrens-Fisher problem and the standardised mean of a normally distributed population. In connection with the first example, the author states that it has still to be investigated whether the " D " method can be used to show the existence of an invariant similar region.

The last half of the paper deals with a criterion for bounded incompleteness in the case of the regular exponential densities as defined above; the completeness of the " D " method in the case of a hypothesis concerning the standardised mean of a normally distributed population and some remarks on the search for the optimum test in this particular problem.

There are three appendices: the first dealing with the detailed proof of a theorem (due to Seidenberg) used in developing the criterion for bounded incompleteness.

(W. R. Buckland)

WITTING, H. (Universität Freiburg, Germany) 5.2 (3.1)

On a χ^2 test with cells determined by order statistics—*In German*

Arch. Math., Karlsruhe (1959) **10**, 468-479 (10 references)

The limit distribution of the classical chi-square statistic for testing the goodness-of-fit is derived by assuming that the cell bounds are determined prior to the realisation of the random variables. This is in contradiction to the usual practical application of the test. Therefore, the author derives asymptotic properties of a chi-square test, when the cell bounds are determined by the $k-1$ order statistics $X_{n,\,j}$ with given ranks $r_{n,\,j}$

$$(r_{n,\,j}-r_{n,\,j-1})/(n+1) = p_{n,\,j} = p_j+o(n^{-1/2}).$$

The test statistic is $T_n = \Sigma\, n[\Pr(S_{n,\,j},\ \hat{\theta}_n)-p_{n,\,j}]^2/p_{n,\,j}$, where $S_{n,\,j} = (X_{n,\,j}, X_{n,\,j-1})$ and $\hat{\theta}_n$ the chi-square minimum estimate of the parameter $\theta = (\theta_1, \ldots, \theta_k)$.

It is shown that $\hat{\theta}_n$ is consistent under general conditions and that the chi-square statistic using this estimate has asymptotically a central (or non-central, respectively) chi-square distribution with $k-s-1$ degrees of freedom. The results are generalised to metric spaces. The cell bounds are constructed in a way similar to the computation of the tolerance limits using statistically equivalent blocks according to the method proposed by Tukey [*Ann. Math. Statist.* (1947) **18**, 529-539].

In the last section, a comparison is made of the asymptotic power of the chi-square test of a simple hypothesis ($H_0: F(x)$) with the power of the Kolmogoroff-Smirnoff test. The Kolmogoroff-Smirnoff test and the chi-square test, when the number k_n of classes tends towards infinite and $p_{n,\,j} = o(1/k_n)$ are consistent against all (fixed) alternative distributions. When a sequence of alternatives with $F(x)+n^{-1/2}g(x)$ is assumed, the power of the Kolmogoroff-Smirnoff test tends towards a value between α and 1, but the power of the chi-square test towards α. In order asymptotically to obtain a power of the chi-square test between α and 1, it is necessary to assume a sequence with $F(x)+{}^4\sqrt{k_n}n^{-1/2}g(x)$.

(E. Walter)

WORMLEIGHTON, R. (Univ. of Toronto, Ontario) 5.6 (5.4)

Some tests of permutation symmetry—*In English*

Ann. Math. Statist. (1959) **30**, 1005-1017 (6 references, 2 tables)

The two-sample sign test is viewed as a test for the permutation symmetry of a bivariate distribution, and is extended to k-variate distributions. Friedman's rank test [*J. Amer. Statist. Ass.* (1937) **32**, 675-701], although originally intended as a substitute for the F-test in randomised blocks, is one such extension. Study of the family of two-sample sign tests obtained by comparing the k co-ordinates pair-wise has yielded a statistic with an asymptotic chi-square distribution from which a further test of symmetry can be constructed. The statistic is based on more degrees of freedom than Friedman's and is sensitive to a greater variety of alternatives. The extension is analogous to that obtained from the Wilcoxon test by Terpstra [*Proc. Kon. Ned. Akad., Wetensh.* A (1954) **57**, 505-512] but in this case, the limiting distribution turns out to be non-singular.

The argument leading to the test is not restricted to the case of complete symmetry but may also be carried through with any specified degree of asymmetry. The coordinates may also be compared m at a time, $2 \leqq m \leqq k$. The argument can be extended and, with a slight modification, includes the derivation of Friedman's test. Thus, a hierarchy of tests of permutation symmetry is constructed: Friedman's rank test corresponds to the case, $m = 1$; when $m = k$, the corresponding test turns out to be Pearson's chi-square.

In this paper the author gives tables to show:

(i) the exact distributions of the symmetric test statistic for $m = 2$, $k = 3$ and sample sizes up to 6.

(ii) a comparison of these exact distributions with the approximating chi-square.

(R. Wormleighton)

ZELEN, M. & JOEL, L. S. (National Bureau of Standards, Washington) **5.2** (3.2)

The weighted compounding of two independent significance tests—*In English*

Ann. Math. Statist. (1959) **30**, 885-895 (9 references, 3 tables, 1 figure)

Two independent F-statistics with the same degrees of freedom for the numerators are available to test a given null hypothesis against a specified alternative. Let F_1 and F_2 be the observed values of F for the two tests, and P_1 and P_2 be the probabilities that the respective F's are greater than or equal to the observed values F_1 and F_2. The authors seek to combine them in a manner suggested by Good for the case where it is assumed that the first test is more powerful than the second. The original idea of combining two tests of significance is due to Fisher. If $0 \leqq \theta \leqq 1$, then the region where " P_1 multiplied by P_2 raised to power θ is less than or equal to C " is proposed as a critical region of the combined test. The null and non-null distributions of this test are derived. The real problem here is to find the value of θ which will minimise the Type II error with respect to the alternative hypothesis in view. Let δ_1 and δ_2 be the non-centrality parameters of the chi-squares in the numerators of the two F's under the alternative hypothesis, where δ_1 is greater than or equal to δ_2. The tables given suggest that the value of θ which achieves this purpose over a wide range of values of other parameters involved is obtained by taking the ratio of δ_2 to δ_1. In general these parameters are not known and have to be estimated from the two individual tests.

(S. S. Shrikhande)

ANDERSON, H. E., Jr. & FRUCHTER, B. (System Development Corp., Santa Monica and Univ. Texas, Austin) **6.2** (6.4)

Some multiple correlation and predictor selection methods—*In English*

Psychometrika (1960) **25**, 59-76 (25 references, 11 tables)

Detailed comparison of three commonly used methods of multiple correlation were made. Methods compared were the Doolittle, Wherry-Doolittle, and Summerfield-Lubin methods of multiple correlation analysis. The Wherry method and the Summerfield-Lubin analysis were shown to be equivalent and their relationship to the Doolittle analysis was indicated. The Wherry and Summerfield-Lubin methods will select the same set of predictor in the same order, provided that the " tests " for terminating the selection procedure is used as a decision tool rather than as an exact test of significance. Selection by the Doolittle method should result in the same set of predictors although the order of selection may differ slightly from the order in the other two methods.

The authors discuss the Cowles-Crout method and the methods employed by Horst. They point out these different methods represent mathematically different approaches for measuring the estimate vector, which is a component of the criterion's vector lying in the predictor vector space. The estimate vector is viewed as colinear with a unit-length vector, which can be written as a linear function of the predictor variables, and the multiple correlation problem reduces to the circumstances of zero-order correlation. The authors recommend the square-root method of multiple correlation because of its ease and compactness and because of the clear interpretation of the coefficients.

(R. E. Stoltz)

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BAHADUR, R. R. (Teachers College, Columbia University, N. York) **6.4** (1.7)

On classification based on responses to *n* dichotomous items—*In English*

USAF School of Aviation Medicine Report 60-13 (1959), 19 pages (2 references)

This report describes a procedure for classifying an individual into one of two groups on the basis of his responses to a given set of dichotomous items together with a procedure for computing the resulting " distance " between the groups. The square root of the Kullback-Leibler information number is used as a plausible distance function. The method relies on an approximation to a given joint probability distribution of response patterns. This approximation is used in constructing the likelihood ratio statistic, which serves as an index for classification. Under reasonable conditions this method serves as an approximation to an optimum classification procedure.

(R. R. Bahadur)

BLACKITH, R. E. (Imperial College Field Station, Sunninghill, Berks, England) **6.3** (6.4)

A synthesis of multivariate techniques to distinguish patterns of growth in grasshoppers—*In English*

Biometrics (1960) **16**, 28-40 (21 references, 3 tables, 1 figure)

This paper is an application of the techniques of multivariate analysis to the studies of the shape and size characteristics of groups of organisms. The latent vectors of a dispersion matrix, representing in this instance the underlying modes of growth of the organism, together with the factors; the canonical variates and the discriminant functions contrasting the differential operation of such modes of growth as have actually been realised in the phenotype; each bring their own contributions to these studies.

Experimental material concerning the multiplicity of polymorphic variations in grasshoppers is presented; and multivariate techniques are applied to distinguish patterns of growth. Grasshoppers of the genus *chorthippus* and of the genera *omocestus* and *stenobothrus* were selected for the experiment, with ten characters, chosen so as to cover the major areas of the body, being measured. A table listing the characters, together with the significant latent and canonical roots, is shown. The extraction of the latent roots and vectors show that the genera *omocestus* and *stenobothrus* exhibit not more than two basically distinct patterns of growth: sexual dimorphism and the symbatic differences of form between species, and between the two colour varieties within the species. Suggestion is made that with such large samples one can detect many of the latent patterns of growth exhibited by these insects even when only a few of these patterns have been elicited sufficiently to produce actual polymorphism in the phenotypes.

Identification of the latent vectors and factors by means of discriminant functions is made, and comparisons of these discriminant functions to distinguish independent or symbatic patterns of growth are shown. Results show that the discriminant function describing sexual dimorphism is essentially the same for many different species and even genera. Sexual dimorphism appears to reflect not only general differences of size but also of certain of the characters. Colour polymorphs differ somewhat in size, but associated distinctions of shape are different from those elicited during the development of sexual dimorphism. This diversity of evolutionary pathways is examined more closely in the canonical analysis.

The demonstration of the existence of a relatively small number of independent patterns of growth in insects seems to justify reservations about the empirical usage of very large numbers of qualitatively appraised characters. Diagrams of taxonomic relationships imply that evolution proceeds in one general direction, and measures the extent to which progress has been made along this generalised evolutionary pathway by a particular taxonomic entity. The burden of this paper is that more than one direction is involved, and that it is at least as useful to know where the process is leading as to know how far it has proceeded.

641 **(W. H. Beyer)**

CONSTANTINE, A. G. & JAMES, A. T. (Commonw. Sci. Industr. Res. Org., Univ. of Adelaide) **6.4** (2.1)

On the general canonical correlation distribution—*In English*

Ann. Math. Statist. (1958) **29**, 1146-1166 (9 references)

This paper is divided into two parts. In the first is given an elementary demonstration of the basic results due to M. S. Bartlett [*Ann. Math. Statist.* (1947) **18**, 1-17]. The method used is the transformation of the canonical correlations and other variables from the original multivariate normal distribution and then integrating-out the (extraneous) variables. The second part of the paper deals with a new method of calculating the conditional moments of any order.

The first part of the paper makes use of the concepts of exterior differential forms and the alternating products of these forms: as discussed by James [*Ann. Math. Statist.* (1954) **25**, 40-75]. An expression is given for the joint distribution of the standard normal variates and for the non-central case. From these, the authors proceed to derive the distributions of the canonical correlation coefficients on the two cases (sections three and four). In the last section of the first part of the paper (section five) the distribution is derived of co-ordinates of random vectors in a random plane: a distribution already used in the development of section three.

The second part of the paper deals with the problem of calculating the conditional moments

$$\mu(t_1, t_2, \ldots, t_p) = \mathscr{E}\left[(s_1^2)^{t_1}(s_2^2)^{t_2}\ldots, (s_p^s)^{t_p}\right]$$

where s_i are the sample correlations between the vectors ξ_i and η_j. These conditional moments are required for the expansion of the distribution of the canonical correlations. In order to avoid the heavy algebra necessitated by computing these moments by direct integration, the device is used of averaging over the orthogonal group. This is outlined in sections seven and eight and followed by a discussion of a device for avoiding complications on the calculations of the moments of invariants: two examples are given here. The final section gives an example of the calculations of the conditional moments—using column vectors instead of row vectors—and the results check with those given by Bartlett.

(W. R. Buckland)

COX, D. R. (Birkbeck College, London) **6.1** (7.6)

Regression analysis when there is prior information about supplementary variables—*In English*

J. R. Statist. Soc. B (1960) **22**, 172-176 (1 table)

Suppose that we are interested in the regression coefficient β of y on x, and that we can observe not only y but also one or more supplementary variables u. Then the observations on u can sometimes be used to obtain an improved estimate of β, provided that we have suitable prior information about the form of the relations among y, x and u. The use of analysis of covariance to obtain adjusted treatment means is a familiar example, where y is the primary variable, x a variable or set of variables representing treatments, and u is the concomitant variable.

In this paper an example is studied where there is simple linear regression of u on x and of y on u, and where given u, y is independent of x. From n triplets (x_i, y_i, u_i), the regression coefficient of y on x is then best estimated from the product of the regression coefficient of y on u, and the regression coefficient of μ on x. The asymptotic variance of the resulting estimate is always less than that of the regression coefficient of y on x, and is appreciably less if y and u, and u and x, both have low correlation.

A practical industrial example where these results may be applicable is when x is a measurement on the raw material before the first stage of processing, or the level of a factor governing the first stage of processing. Suppose also that the corresponding value of y is a measure of the property of the final product, and that u is a measure of a property of the output from the final stage. Under some circumstances it may then be reasonable to postulate the model of the previous paragraph.

Implications of this for experimental design are noted, and it is pointed out that much more general situations can be considered.

(D. R. Cox)

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DHONDT, A. (Centre d'Études de Recherche Opérationnelle, Brussels) **6.6** (-.-)

On a generalisation of a theorem of R. Frisch in confluence analysis—*In French*

Cahier du Centre d'Études de Rech. Opérat. (1960) **2**, (page numbers unobtained)

In his publication *Statistical Confluence Analysis by means of Complete Regression Systems* (1934), Frisch proposed new methods to find limits to the structural equations in the sets of variables which are found to be admissible.

One of the leading ideas on which his technique is based, is included in the following theorem:

> " The structural vector of a set of n variables is confined to the angle formed by the elementary regression vectors."

In the proof of this theorem, Frisch confined himself to an example in two variates. The theorem has been generalised by Reiersöl who has given a very complicated argument introducing instrumental variables.

The present publication is concerned with a direct proof of the generalised theorem inspired by Reiersöl's demonstration. We first derive upper limits for the disturbing intensities: then we formulate the fundamental hypothesis required for the exactness of the theorem. This hypothesis states that the elementary regression vectors should have compatible signs, that is the straight lines in which the vectors are lying should be confined to two diametrically opposite orthants. The theorem can then be set forth as follows:

> " If the elementary regression vectors have compatible signs, then the structural vector is confined to the angle of these vectors after they have been sign-corrected in such a way that they will lie in the same orthant."

(A. Dhondt)

EEDEN, CONSTANCE van & KORSWAGEN, RINA (Mathematical Centre, Amsterdam) **6.5** (5.6)
Manual for Kendall's rank correlation test—*In Dutch*
Statist. Dept., Math. Centre, Report S262 (1960)

This report contains a description of Kendall's rank correlation test, illustrated by means of a large number of examples. The well-known facts about this test are dealt with in an elementary way. One section is devoted to the method of Sandiford & Holmes, as discussed by Griffin [*J. Amer. Statist. Ass.* (1958) **53**, 441], for the graphical computation of the test statistic S: a clear proof of the method is given.

The report contains the following tables:

1 and 2. Exact probabilities $\Pr[S \geqq S]$ for $n = 4(1)40$.

3. Exact upper critical values of S for $n = 4(1)40$, and (one-sided) = 0·005; 0·01; 0·025; 0·05; 0·10.

4. Approximations to the upper critical values of S for $n = 41(1)100$ and the same levels as in table 3.

5. Values of $\sigma^2 = 1/18\{n(n-1)(2n+5)\}$, and of σ, for $n = 4(1)100$.

6. Table of t^2 and t^3 for $n = 4(1)40$ (facilitating the calculation of σ^2 in the cases where ties occur).

7. Table of the normal distribution.

This report is the third one in a series of manuals, the other two being on Wilcoxon's two sample test (S 176), and on Wilcoxon's test for symmetry (S 208).

(A. R. Bloemena)

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FINNEY, D. J. (University of Aberdeen) **6.8** (9.0)
The design and analysis of an immunological assay—*In English*
Acta Microbiol. Acad. Sci. Hung. (1959) **6**, 341-368 (20 references, 9 tables, 1 figure)

The estimation of relative potency of vaccines, by reference to the proportion of test animals protected from standard challenge dose of infective organisms, is discussed. The estimates and fiducial limits, if any, of the following methods are compared: inspection, interpolation, Dragstedt-Behrens, Spearman-Kärber, graphical normal equivalent deviate, graphical logit, normal equivalent deviate (maximum likelihood), logit (maximum likelihood), logit (minimum χ^2), angle (maximum likelihood). The first four are not recommended by the author; the next three only for rapid approximate evaluation. For general use the normal equivalent deviate angle or logit (either maximum likelihood or minimum χ^2) are recommended.

(I. Juvancz)

GIBSON, W. A. (Personnel Research Branch, Adjutant General's Office, Washington) **6.3** (0.6)

Remarks on Tucker's inter-battery method of factor analysis—*In English*

Psychometrika (1960) **25**, 19-25 (2 references, 3 tables, 1 figure)

This paper presents a discussion of the implicit assumptions made by Tucker in developing the inter-battery method of factor analysis. The effects of violations of these assumptions are discussed and a suggested modification of the inter-battery method to avoid these restrictions is considered.

Let the intercorrelation matrix for battery one be identified as $\mathbf{R}_{11}$, the inter-correlation matrix for battery two as $\mathbf{R}_{22}$, and the correlations between battery one and battery two are R_{12} or $\mathbf{R}_{21}$, one being the transpose of the other. The inter-battery method may be applied by forming two symmetric matrices, $\mathbf{H}_1$ must equal $\mathbf{R}_{12}\mathbf{R}_{12}'$ and $\mathbf{H}_2$ must equal $\mathbf{R}_{12}'\mathbf{R}_{12}$, obtaining their characteristic roots and vectors. The non-vanishing characteristic roots of $\mathbf{H}_1$ and $\mathbf{H}_2$ are identical, if they are represented in a diagonal matrix $\mathbf{\Lambda}^2$. The first diagonal element in $\mathbf{\Lambda}$ must be taken as the negative square root of the corresponding element in $\mathbf{\Lambda}^2$ in order to satisfy the equation $\mathbf{R}_{12} = \mathbf{A}_1\mathbf{A}_2'$. It then follows that the first diagonal element in $\mathbf{\Lambda}^{\frac{1}{2}}$ is imaginary because it is the square root of a negative number. Consequently the first column of $\mathbf{A}_1$ and of $\mathbf{A}_2$ must contain imaginary factor loadings in order that equation three may hold.

The author points out that in addition to this appearance of imaginary factor loadings, some negative communalities are implied by $\mathbf{A}_1$ and $\mathbf{A}_2$, and some residuals may exceed unity, while other residuals are alarmingly large. The author points out that the inter-battery method treats $\mathbf{R}_{12}$ as defining a space in which the two vector configurations have the same principal components reference frame and the same sums of square projections thereon. That space may be complex or otherwise different from the factor space of overlapping vector configurations.

Tucker's method of presenting the inter-battery method, in which $\mathbf{A}_1$ and $\mathbf{A}_2$ are treated as orthogonal factor matrices for the overlapping factor space assumes implicitly that, within that factor space, the two sets of principal axes and the associated characteristic roots coincide. If these assumptions are not met, then $\mathbf{A}_1$ and $\mathbf{A}_2$ will exhibit such properties as non-Gramian and poor reproduction of $\mathbf{R}_{11}$ and $\mathbf{R}_{22}$.

An appropriate pair of orthogonal matrices, $\mathbf{F}_1$ and $\mathbf{F}_2$, might be obtainable from $\mathbf{A}_1$ and $\mathbf{A}_2$ by means of the equations, $\mathbf{F}_1 = \mathbf{A}_1 T$ and $\mathbf{F}_2 = \mathbf{A}_2(T^{-1})'$, where T is a linear transformation. The suggested method of solution for T is to have it be such that

$$\mathbf{F}_1\mathbf{F}' = \mathbf{A}_1 TT'\mathbf{A}_1' \cong \mathbf{R}_{11}$$

$$\mathbf{F}_2\mathbf{F}_2' = \mathbf{A}_2(T^{-1})'T^{-1}\mathbf{A}_2' = \mathbf{A}_2(TT')^{-1}\mathbf{A}_2' \cong \mathbf{R}_{22}$$

from which least square expressions for TT' and $(TT')^{-1}$ could be obtained by the usual techniques.

(R. E. Stoltz)

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GOODMAN, L. A. (University of Chicago) **6.5** (-.-)

Partial tests for partial taus—*In English*

Biometrika (1959) **46**, 425-432 (12 references)

Considering three variables P, Q, R, partial correlation coefficients are used in the study of the question whether an apparent association between Q and R is really due to the association of each with P, when the data consists only of n trials of integers (P_i, Q_i, R_i) giving the ranks of the observations.

The null hypothesis considered in this paper is that, conditionally on P_i being held fixed, the corresponding Q and R are independent. It may be noted that this differs from the usual definition of conditional independence of random variables; the conditions it implies on the joint distributions depend on n. These points are not discussed. It is further assumed that the probability that two Q's are concordant with the corresponding P's (conditional on the P-ranks) depends only on the difference between the P-ranks; and similarly for the R's. Some situations where these assumptions are appropriate are to be described in a forthcoming paper in *J. Amer. Statist. Ass.*

It is shown that a series of $n-1$ partial rank correlation coefficients can be defined by counting concordances of pairs of Q's and R's with the corresponding P's; firstly for pairs of P's with adjacent ranks, secondly for pairs of P's with ranks differing by two, and so on. The counts can be displayed in a series of fourfold tables. Kendall's partial tau (τ) can be calculated from the fourfold tables obtained by summing corresponding entries in the above table; it appears that this overall table may indicate dependence even when there is complete independence in the component table.

Large-sample results are given concerning the asymptotically normal distributions of these partial coefficients under the specified null hypothesis. It is stated that these partial coefficients are related to, but differ from, those introduced by Somers [*Biometrika* (1959) **46**, 241-246: abstracted in this journal, No. 255, **6.5**].

(C. L. Mallows)

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GOODMAN, L. A. & KRUSKAL, W. H. (University of Chicago) **6.9** (11.9)

Measures of association for cross classifications. II. Further discussion and references—*In English*

J. Amer. Statist. Ass. (1959) **54**, 123-163 (157 references)

This paper is supplementary to the 1954 publication of the authors in the same journal [*J. Amer. Statist. Ass.* (1954) **49**, 723-764]. The earlier work is not repeated or summarised in the present paper.

A brief introduction and summary is followed by a section extending the discussion of measures of association presented in the 1954 paper. Several topics are treated including measures useful when diagonals are not of interest, a relation between Yule's Y and other (" lambda ") measures, a measure based on proportional prediction, measures of association between subsets of categories and comparison of degree of association in two cross classifications. Two minor corrections to the 1954 paper are noted.

The remainder, constituting the major portion, of the paper greatly extends the historical and biographical material on cross classifications. In a section on late nineteenth- and early twentieth-century work a starting point is taken with the efforts of American and German scientists, notably Doolittle, Peirce & Koppen, to measure predictive success in meteorology. The use of measures of association by Körösy to study the effectiveness of smallpox vaccination is described, together with related work of Jordan & Quetelet. Subsections are devoted to Benini and attraction or repulsion measures; Lippo, Tönnies' measure for square cross classification with ordered categories; Deuchler, and the extensive work of Gini and the Italian school.

The final and largest section of this paper is concerned with more recent publications; the first subsection of this refers to several textbook presentations. Reliability measures, those in which both polytomies are the same and refer to two methods of assignment, are introduced with the work of Wood, Reuning, Cartwright and others. Measures that are zero if, and only if, there is independence are set forth and publications of Cramér, Steffensen, Pollaczek-Geiringer, Höffding, Eyraud, Fréchét and Féron are discussed. A measure of dissimilarity in a by 2 cross classifications suggested by Gini and developed by Florence, Hoover, Duncan & Duncan and Bogue is disussed, and other such measures due to Boas, Long and Loevinger are indicated. Short subsections are given to measures based on Lorenz or cost-utility curves and on Shannon-Wiener information. Another subsection lists more recent contributions of the Italian school, largely based on the early work of Gini. Papers from the medical literature concerned with problems of inference from one population to another are cited. A longer subsection is devoted to measures based on latent structures with particular reference to the work of Lazarsfeld and Kendall. Subsections deal with recent applications in meteorology, in ecology and in anthropology. The last subsection lists other suggestions of Harris, Irwin, Lakshmanamurti, and Smith.

The references are extensive and the great majority of them are additional to those given in the 1954 paper of the authors or in the related 1958 paper of Kruskal.

649 (S. Krane)

HEWLETT, P. S. & PLACKETT, R. L. (Pest Infestation Lab., Slough, and Liverpool Univ.) **6.8** (11.1)

A unified theory of quantal responses to mixtures of drugs : non-interactive action—*In English*

Biometrics (1959) **15**, 591-610 (29 references, 1 table, 3 figures)

In a previous paper [*J. R. Statist. Soc.* B (1952) **14**, 141-163] the authors considered models of the joint action of a pair of drugs. The present paper presents a unified theory for similar and dissimilar joint action in the absence of interaction.

Without loss of generality, presentation is in terms of normal equivalent deviates. Conditions for non-response in the individual organism for every form of non-iterative joint action is inferred. These conditions are superimposed upon a bivariate normal distribution of tolerances for the two drugs. Consistent mathematical models that permit any degree of correlation, including negative correlation, between the biochemical or physiological actions of the two drugs are developed. Non-parallelism between the normal equivalent deviate—log-dose lines, which has been observed experimentally for similar drugs, is provided for in the developed models and is given a plausible interpretation.

As a prelude to the development of the models detailed consideration is given to the action of a single drug. In particular the relationship between the dose and the amount of the drug actually reaching the site of action is discussed. A " penetration parameter " is defined and it is pointed out that this parameter can differ for different similar drugs and thus lead to different slopes.

Correlation of the tolerances between two similar drugs can be incomplete if, for example, the two drugs have the same site of action but one and not the other is metabolised to an inactive substance at a site of loss; the tolerance would then depend partly on the action tolerance and partly on the capacity of this site of loss. Strong positive correlations can result even if sites of action are different and negative correlations can occur if the same enzyme system promotes the metabolism of one drug to a more active metabolite and the other to a less active metabolite.

General equations for simple similar joint action and for independent joint action are given in terms of ratios of amount acting to amount acting required for response. Two examples of general models, giving rise to these extreme special cases, are considered. In both examples a single parameter indicates the degree of similarity or dissimilarity in the joint action. A discussion of the determination of the probability of non-response in terms of the bivariate normal distribution of tolerances is given. Dose-response curves for certain mixtures of drugs have been worked out and are discussed. Finally, earlier models by several workers are interpreted as special cases of the general model presented here.

(M. E. Turner)

JOHNS, M. V. (Teachers College, Columbia University, N. York) **6.4** (1.8)

An empirical Bayes approach to non-parametric classification—*In English*

USAF School of Aviation Medicine Report 60-16 (1959) 21 pages (5 references)

This report considers the problem of classifying an individual into one of two categories in the non-parametric case. Classification procedures are proposed which make use of all observations previously taken on individuals independently selected from the same population. These procedures have risks which are asymptotically optimum as the number of prior observations become large. The loss due to misclassification is assumed to depend on the value of a random variable associated with the individual but not observed at the time of classification. The case where only one of the losses due to misclassification of individuals previously selected from the population is known is also considered, and similar results are obtained.

(M. V. Johns)

651

KITAGAWA, T. (Kyushu University) **6.1** (9.3)

Successive process of statistical inference applied to linear regression analysis and its specialisation to response surface analysis—*In English*

Bull. Math. Statist. (1959) **8**, 80-114 (15 references)

This paper treats the successive and sequential process in linear regression analysis when the estimation after preliminary tests of significance plays a fundamental role. First, the author discusses biased estimation of linear regression coefficients under an incompletely specified model and gives expressions for the estimated coefficients and sum-of-squares of residuals under the complete model. The sampling distributions associated with this biased estimation are also explained.

Secondly, two and three stage procedures of estimation of the regression coefficients after a preliminary test of significance are discussed in the case where some hypotheses concerning the coefficients are preliminarily tested and estimation is made in the respective manner according to acceptance and rejection of the hypotheses. The author extends this idea to the sequential procedure, where the experimenter decides by test of significance whether the further experiments are to be performed in order to obtain more data for a better fit as linear regression or the experiment may be stopped and the linear regression may be estimated on the basis of the data already obtained.

The latter part of the paper is mainly concerned with applications of the results mentioned above to the response surface analysis developed by Box and his colleagues. The rotatable designs of order two and three in the two-dimensions, in the sense of Box-Hunter, are treated and various expressions necessary for practical performances based on these designs are given. Throughout this paper, the mean-square error of the estimated regression function is considered as the measure of accuracy for each estimation procedure and then the overall distance function is defined over a given domain by means of a given weight function. Finally evaluation of probabilities of the events, and of mean values of the random variables associated with various procedures of estimation considered in this paper are discussed.

(M. Siotani)

652

KONIJN, H. S. (University of Sydney) **6.0** (6.6)
Models used in statistical studies of relations between uncontrolled variables—*In English*
Aust. J. Statist. (1959) **1**, 82-93 (28 references)

This paper reviews the recent work of some twenty authors on models of statistical relationships between variables which are not under the control of the investigator. Six basic models are examined:

(i) the joint distribution model;
(ii) the exact linear structural relation model;
(iii) the conditional distribution model with equation error;
(iv) the conditional distribution model with errors in observations;
(v) the mean-square regression model; and
(vi) the random coefficient model.

Particular attention is paid to assumptions under which one can find unbiased or consistent estimates of the coefficients of the relations between variables. In view of the wide choice of models available, the author expresses the hope that " Perhaps the present paper—together with the literature referred to—will be of some aid in selecting appropriate models and methods of statistical inference in specific cases and in avoiding pitfalls".

(E. W. Bowen)

653

KUPPERMAN, M. (George Washington University, Washington, D.C.) **6.9** (5.6)
A rapid significance test for contingency tables—*In English*
Biometrics (1959) **15**, 625-628 (8 references)

The simple and rapid calculation of a statistic for testing the independence between the rows and columns of a contingency table has recently been facilitated by two publications. A table of $2n \ln n$ for n (1)2009—4 decimal places—by Woolf [*Ann. Hum. Genet.* (1957) **21**, 397-409] and a table of $n \ln n$ for n (1)1000—10 decimal places—by Kullback [*Information Theory and Statistics* (1959). New York: Wiley] makes this possible.

The approach used by Woolf is the likelihood ratio whereas Kullback uses information theory. The information statistic $2\hat{I} = -2 \ln(\text{likelihood ratio})$ and the former gives a reasonably close approximation to χ^2 in large samples. For this reason its use avoids much of the tiresome calculations associated with the use of χ^2 in contingency tables.

An example is given and it is also suggested that either of the two tables referred to above could be usefully extended so as to deal with the frequencies greater than a thousand which often arise in practice.

(W. R. Buckland)

654

LORD, F. M. (Educational Testing Service, Princeton) **6.3** (7.3)

Some relations between Guttman's principal components of scale analysis and other psychometric theory—*In English*

Psychometrika (1958) **23**, 291-296 (14 references, 1 table)

Guttman's principal components for the weighting system are the item scoring weights that maximise the generalised Kuder-Richardson reliability coefficient. The principal component for any item is effectively the same as the factor loading of the item divided by the item standard deviation, the factor loadings being obtained from an ordinary factor analysis of the item intercorrelation matrix.

After reviewing Guttman's definitions of a principal component of the score system and of the weighting system and certain properties of the system, the author shows that:

(i) Guttman's principal components for the weighting system are the same as the sets of weights that will maximise the generalised Kuder-Richardson reliability coefficient.

(ii) Guttman's principal components for the weighting system (and thus the scoring weights for maximising test-reliability also) are effectively the same as certain sets of item weights obtained by factoring the matrix of item intercorrelations.

(iii) Guttman's principal components for the scoring system are perfectly correlated with the scores obtained for the examinees when the items are weighted so as to maximise the generalised Kuder-Richardson reliability coefficient.

These results are believed by the author to be new, since Guttman's method of obtaining the principal components requires the factoring of a matrix with n/m as many rows and columns as does the present method.

(R. E. Stoltz)

NEISWANGER, W. A. & YANCEY, T. A. (University of Illinois, Urbana) **6.6** (10.7)

Parameter estimates and autonomous growth—*In English*

J. Amer. Statist. Ass. (1959) **54**, 389-402 (8 tables)

Monte Carlo techniques are used by the authors in this paper, the purpose of which is to investigate the effect of one kind of specification error. A two-equation over-identified system is constructed wherein both the shocks and the exogenous variables contain linear time trends.

Parameters are estimated by both limited-information and least-squares techniques ignoring the presence of the trends. The properties of the sampling distributions are investigated and compared with the distributions of estimates obtained when time is included as an exogenous variable. The test for independence in the residuals is likewise investigated. It is demonstrated that the addition of time as an exogenous variable eliminates the bias in the limited-information estimates.

(W. A. Fuller)

PERKAL, J. (University of Wroclaw) **6.4** (10.9)

The notion of likeliness related to population—*In Hungarian*

Magy. Tud. Akad. III *Oszt. Közl.* (1959) **9**, 423-426 (4 references)

The author proposes a notion of likeness which is symmetrical, transitive and broader than the equality. Population individuals x are characterised by points of a multi-dimensional space the respective properties being the coordinates. The transformation $S(x)$ is called " likeness-preserving " with respect to the set family S if it does not separate off the x-containing member of S. The invariants of $S(x)$ are named the indices of the actual likeness. A set of indices covariant with likeness is a " complete index system ". Further, x and y are " alike in the deviation sense ", if the straight line through them contains the centroid. They are " alike in the natural sense " if this line is parallel to the vector composed by the respective standard deviations. This latter notion has been applied to studies on sportsmen.

(J. Fischer)

RÉNYI, A. (Math. Inst., Hungarian Academy of Sciences, Budapest) **6.0** (6.9)

On measures of dependence—*In English*

Acta Math. Acad. Sci. Hung. (1959) **10**, 441-451 (10 references, 1 figure)

After some introductory remarks on definitions and notations, the author proposes seven postulates for a measure $\delta(\xi, \eta)$ of interdependence between the random variables ξ and η, as follows:

A. $\delta(\xi, \eta)$ is defined for any pair of random variables ξ and η, neither of them being constant with probability 1.

B. $\delta(\xi, \eta) = \delta(\eta, \xi)$.

C. $0 \leqq \delta(\xi, \eta) \leqq 1$.

D. $\delta(\xi, \eta) = 0$ if and only if ξ and η are independent.

E. $\delta(\xi, \eta) = 1$ if there is a strict dependence between ξ and η, i.e. either $\xi = g(\eta)$ or $\eta = f(\xi)$ where $g(x)$ and $f(x)$ are Borel-measurable functions.

F. If the Borel-measurable functions $f(x)$ and $g(x)$ map the real axis one-to-one on to itself, $\delta[f(\xi), g(\eta)] = \delta(\xi, \eta)$.

G. If the joint distribution of ξ and η is normal, then $\delta(\xi, \eta) = |R(\xi, \eta)|$ where $R(\xi, \eta)$ is the correlation coefficient of ξ and η.

The table shows whether the above postulates are fulfilled by some measures of dependence:

	A/	B/	C/	D/	E/	F/	G/
Correlation coefficient:	−	+	+	−	−	−	+
Correlation ratios:	−	−	+	−	−	−	+
Max $[\theta_\xi(\eta), \theta_\xi(\xi)]$:	−	+	+	−	+	−	+
Maximal correlation:	+	+	+	+	+	+	+
Mean square contingency:	−	+	−	+	−	+	−
$\Gamma(\xi, \eta) = C(\xi, \eta)/\sqrt{1+C^2(\xi, \eta)}$:	−	+	+	+	−	+	+

The mean square contingency, $C(\xi, \eta)$ is defined by Rényi in his paper " New version of the probabilistic generalisation of the large sieve " [*Acta Math. Acad. Sci. Hung.* (1959) **10**, 218-226: abstracted in this journal, No. 252, **6.0**].

Later the author considers the conditions under which the maximal correlation can be attained; that is to say, the defining supremum is also maximum. Denoting the conditional expected value by $M(\xi \mid \eta)$ he points out that the relation

$$Af = M\{M[f(\xi) \mid \eta] \mid \xi\}$$

defines a linear, bounded, self-adjoint and positive definite operator A which transforms the Hilbert-space of all random variables $f = f(\xi)$ with zero means and finite variances into itself. In the first theorem of this paper it is proved that complete continuity of A is sufficient to attain $S(\xi, \eta)$, and in the second theorem the presence of this complete continuity is shown to follow from $C(\xi, \eta) < \infty$.

Finally, the author shows that, in the case when maximal correlation is attainable, there exists real functions $f_0(x)$ and $g_0(x)$ such that $M[f_0(\xi) \mid \eta] = \delta(\xi, \eta) g_0(\eta)$ and $M[g_0(\eta) \mid \xi] = S(\xi, \eta) f_0(\xi)$. Further, he gives an example where $S(\xi, \eta) = 1$ is not attained.

(J. Fischer)

SARGAN, J. D. (University of Leeds) **6.6** (10.7)

Estimation of relationships with autocorrelated residuals by the use of instrumental variables—*In English*

J. R. Statist. Soc. B (1959) **21**, 91-105 (11 references)

This present paper can profitably be read in conjunction with an earlier paper by the same author entitled " On the estimation of economic relationships by means of instrumental variables [*Econometrica* (1959) **26**, 393-415: abstracted in this present journal, No. 90, **6.6**] since the method of estimation put forward there is extended and generalised. It is desired to estimate linear relationships between variables which are subject to error, in this case where the residuals are in the autocorrelation scheme of lag one.

The relationship is then considered as connecting a set of variables which has coefficients which are functions of a set of parameters less in number than the number of variables in the relationship. The " instrumental variables " method of estimation is applied and the asymptotic error-variance matrix of the estimates is obtained. Tests for the indentificability of the estimates and for their existence are suggested (asymptotic only), and confidence regions based on the asymptotic results are given. A comparison with maximum likelihood estimates is made.

(Florence N. David)

WAGNER, H. M. (Stanford University, California) **6.1** (0.8)

Linear programming techniques for regression analysis—*In English*

J. Amer. Statist. Ass. (1959) **54**, 206-212 (12 references)

Two types of problems are presented by the author in this paper and he solves them by the linear programming technique:

(*a*) Minimising the sum of absolute deviations for a regression model which is linear in the parameters.

(*b*) Minimising the maximum absolute value (Tchebysheff-criterion) for the same model.

For both (*a*) and (*b*) the dual relationship is employed to reduce the number of restrictions. For (*a*) with p regression coefficients and k observations, the bounded variable algorithm is introduced to reduce the problem to one with p restrictions. The procedure with $p+k$ restrictions is also given for the more familiar original and revised simplex methods. Concerning (*b*) the author shows that $p+k+1$ restrictions are required in the original or revised simplex methods. A numerical example, with $p = 1$, is given for both (*a*) and (*b*).

(W. T. Lewish)

WILLIS, R. H. (Carnegie Institute of Technology, Pittsburg) **6.2** (6.3)

Lower bound formulas for the mean inter-correlation coefficient—*In English*

J. Amer. Statist. Ass. (1959) **54**, 275-280 (4 references)

It is the purpose of this paper to show how small the inter-correlations of a set of variables can be on the average. The author considers this lower bound, $\bar{\rho}_{LB}$, under three conditions:

(i) No restrictions on the value of the correlations (i.e. $1 \geqq \rho_{\beta_i \beta_j} \geqq -1$).

(ii) No two variables can correlate negatively (i.e. $1 \geqq \rho_{\alpha_k \alpha_l} \geqq 0$). This condition might be pertinent to a psychologist dealing with a matrix of correlations resulting from measures of mental abilities each of which, according to certain theories, measure to some extent a general trait in addition to other traits.

(iii) There are sets of variables, α sets, included in the total number of variables, which cannot correlate negatively among themselves

[i.e. $(1 \geqq \rho_{\alpha_k \alpha_l} \geqq 0)$, $(1 \geqq \rho_{\alpha_k \beta_i} \geqq -1)$, and $(1 \geqq \rho_{\beta_i \beta_j} \geqq -1)$].

This condition also might be pertinent to the psychologist dealing with correlations among variables, some of which are measures of mental abilities.

The author uses a geometrical argument to show that under (i) $\bar{\rho}_{LB} = -1/N-1$ when the total number of variables, N, is even; and $\bar{\rho}_{LB} = -1/N$ when N is odd. The argument uses the relationship $\rho_{ij} = \cos \phi_{ij}$ and points out that $\bar{\rho}_{LB}$ will have its minimum value when $\Sigma \phi_{ij}$ has its maximum value.

Continuing his geometrical argument the author shows that under (ii) $\bar{\rho}_{LB} = 0$ when the rank of the matrix, **n**, is N and

$$\bar{\rho}_{LB} = (N-n)\rho + pq/N(N-1)$$

when $n<N$, where $N = np+q$, so that there are $(n-q)$ vector clusters of size p and q clusters of size $(p+1)$, $0 \leqq q \leqq \min(n-1, N)$; the within-cluster correlations will be $+1{\cdot}00$ and the between cluster correlations zero.

Under (iii) the equations for (i) hold if the number of variables which can correlate negatively among themselves, N_β, exceed or equal the number of variables that can not correlate negatively among themselves, N_α. If $N_\alpha > N_\beta$, then

$$\bar{\rho}_{LB} = np(p-1) + 2(pq - N_\beta)/N(N-1),$$

where $N = 2N_\beta$ α-vectors are divided into $(n-q)$ clusters of size p and q clusters of size $(p+1)$, $0 \leqq q \leqq \min[(n-1), (N-2N_\beta)]$.

(L. Wolins)

661

BIRNBAUM, A. (New York University) **7.9** (9.2)

On the analysis of factorial experiments without replication—*In English*

Technometrics (1959) **1**, 343-357 (8 references, 5 tables, 1 figure)

Inferences from factorial experiments without replication are usually based on a formal assumption that certain interactions are zero. In a research situation which is completely exploratory, any statistical model giving a formal basis for informative inferences will generally be too schematic and restrictive of unknown conditions to be claimed " valid," or a basis for inferences which are " valid " except in the hypothetically formal sense. A model of this kind is, perhaps along with other models, a basis for " plausible inferences," i.e. inferences drawn in a formally-valid manner, based on a model which is more or less plausible. Under some conditions (which are reviewed), the following schematic model is useful and plausible: the m contrasts a_i are independent, normal, homoscedastic; at most (any) r of their means are non-zero. For $r = 1$, to decide which, if any, mean is non-zero, the statistic $\max_i a_i^2/\Sigma_i a_i^2$ is optimal. An alternative graphical procedure developed by Daniel [*Technometrics* (1959) **1**, 311-341; abstracted in this present journal, No. 664, **7.7**] which has important advantages, is related to the ratio of $\max_i | a_i |$ to another ordered $| a_i |$. Critical values, power and related properties, comparisons with more conventional statistics, and discussion of cases $r > 1$, are given.

(A. Birnbaum)

663

DANIEL, C. (New York City) **7.7** (9.2)

Use of half-normal plots in interpreting factorial two-level experiments—*In English*

Technometrics (1959) **1**, 311-341 (17 references, 8 tables, 19 figures)

The empirical cumulative distribution of the well-known set of orthogonal contrasts computed from a 2^p factorial experiment is plotted on a " half-normal " grid that linearises the expected positions in a null experiment. Wide deviations of the largest contrasts from the pattern set by the smaller ones permit judgments of the reality of the corresponding effects. Wide deviations of the smaller contrasts from the pattern of a straight line through the origin permit judgments of certain types of defects in the data. A single bad value, two bad values, inadvertent plot-splitting, and error-variance dependence on mean value, each produces a characteristic modification in the shape of the plot. Numerical examples are given.

(C. Daniel)

664

GAUTSCHI, W. (Indiana University, Bloomington) 7.2 (5.1)

Some remarks on Herbach's paper " Optimum nature of the *F*-test for Model II in the balanced case "—*In English*

Ann. Math. Statist. (1959) **30**, 960-963 (8 references)

This paper presents a lemma which shows that, in Herbach's paper referred to in the title, [see *Ann. Math. Statist.* (1959) **30**, 939-959: abstracted in this present journal, No. 666, **7.2**], the standard *F*-test for the absence of interaction in a two-way balanced classification is a uniformly most powerful similar test. The lemma also shows that, in a balanced Model II design (with the usual normality assumptions), the standard estimates of the variance components are unbiased and of minimum variance.

(S. S. Shrikhande)

665

HERBACH, L. H. (New York University) **7.2** (5.1)

Properties of Model II-type analysis of variance tests : A. Optimum nature of the *F*-test for Model II in the balanced case—*In English*

Ann. Math. Statist. (1959) **30**, 939-959 (18 references, 1 table)

A distribution analogous to the canonical distribution used in testing the general linear hypothesis is developed for Model II (components-of-variance model) analysis of variance for balanced classifications. The author uses the standard form to show that the *F*-test is uniformly most powerful: this is similar to testing for the non-existence of main effects in the balanced one or two-way classification. In the balanced one-way classification the likelihood ratio test is not the *F*-test, but for purposes of testing we can act as if it were. In the case of two-way balanced classification, however, not only is the likelihood ratio test not the *F*-test, but (unlike the previous case) is not even equivalent to it for small levels of significance.

The author also shows that the uniformly most powerful invariant test for the balanced one-way classification is the *F*-test. In the case of two-way balanced classification there may not be any uniformly most powerful invariant test. For testing the absence of a main effect in the case of the two-way balanced classification it is shown that, of all the invariant tests whose power is independent of nuisance parameters, the *F*-test is the most powerful. The methods of this paper can be generalised to the multiple classification case and especially to multifold nested classifications which are very useful in sampling theory.

(S. S. Shrikhande)

666

HUGHES, H. M. & DANFORD, M. B. (School of Avn. Med., Brooks Air Force Base, Texas) 7.3 (9.7)
Repeated measurement designs, assuming equal variances and covariances—*In English*
USAF School of Aviation Medicine Report 59-40 (1958) 19 pages (4 references, 1 table)

This paper is concerned with the analysis of measurements taken at fixed times on the same individuals. On each group of individuals of size n_i, treatment i is given ($i = 1, 2, \ldots K$). The problem of testing for fixed time effects and group (or treatment) by time interaction is presented in the light of Scheffé's consideration of the two-way mixed model. It is shown that if equal variances and covariances are assumed for this repeated measurement (or extended two-way mixed model) design, the commonly used tests of significance are appropriate.

(M. B. Danford)

667

MANDEL, J. (National Bureau of Standards, Washington, D.C.) 7.3 (7.5)
The measuring process—*In English*
Technometrics (1959) **1**, 251-267 (6 references, 6 tables)

This paper deals with the theory of a newly proposed method for the statistical study of measuring processes. The practical aspects, including computational details, are discussed in a companion paper published in the *ASTM Bulletin*. In the present paper a theoretical framework is developed for the mathematical expression of the sources of variation in measuring methods. Essentially, a linear model is presented for the expression of measurements classified: (i) according to internally-homogeneous groups, such as those formed by measurements obtained in a single laboratory during a single testing session, and (ii) according to the magnitude of the property under measurement. The latter classification generally coincides with that commonly called " materials " or " samples ".

Referring to a " cell " as the set of replicate measurements obtained by each group (laboratory) for each material, it is proposed to start the analysis by examining the within-cell variances, in terms of their dependence on the magnitude of the measurement. If a systematic trend is found in these variances, a transformation is made prior to further analysis of the cell averages. Representing a cell-average (in the transformed scale, if necessary) by Z_{ij}, the linear model is expressed by a set of relations of the type:

$$Z_{ij} = \mu_i + \beta_i(x_j - \bar{x}) + \eta_{ij}$$

where μ_i and β_i are parameters characteristic of the " laboratory ", x_j a value characteristic of the " material ", and η_{ij} a random variable. It is found that by selecting x_j as the average $Z_{.j}$, an appropriate analysis of variance can be made in terms of the model, and the parameters can be estimated by regression methods. The quantity commonly referred to as " Laboratory × Material interaction " is composed of a systematic portion, expressing the variability among the slopes β_i of the straight lines representing the " laboratories " in the model, and a random part, η_{ij}. The latter is in turn partitioned into the error of replication ϵ_{ij}, and another random component, λ_{ij}. This additional component can be related to the disturbances caused by interfering substances or properties.

From the analysis of variance, expanded in this fashion, components of variance can be derived for the characterisation of the various sources of variability. The contribution of each laboratory to the variance of the random components η_{ij} is separately computed, allowing the detection of " laboratories " that are unduly erratic. An illustration of the proposed method, based on data taken from the chemical literature, is appended.

(J. Mandel)

MANDEL, J. (National Bureau of Standards, Washington, D.C.) **7.5** (7.1)
The analysis of Latin squares with a certain type of row-column interaction—*In English*
Technometrics (1959) **1**, 379-387 (6 references, 6 tables)

A serious limitation in the use of Latin squares is the confounding of the main effect of each factor with the interaction of the remaining two factors. In some cases, the interaction of rows and columns can be expressed as a multiplicative term of assigned factors associated with rows and columns. Consider a Latin square of n rows, n columns and n treatments. Representing these categories by i, j, and k, respectively, consider the model:

$$y_{ijk} = \mu + \rho_i + \gamma_j + \beta A_i B_j + \theta_k + \epsilon_{ijk}$$

where the last term is a random error. It is often possible to assign to the n rows, quantities A_i, such that $\sum_i A_i = 0$, and to the n columns, quantities B_j such that $\sum_j B_j = 0$, and to assume that the interaction between rows and columns is adequately expressed by the term $\beta A_i B_j$. The complete analysis of variance is presented in terms of this model. All relevant tests of significance are discussed. The estimation of β and of its standard error is given.

An extension of the model is also presented for cases in which the treatment effects θ_k can be considered as linearly related to a concomitant variable u_k, such that $\theta_k = K(u_k - \bar{u})$.

The procedure is illustrated by means of an example taken from a study of the abrasion resistance of leather. The comparison of the proposed method with other tests for non-additivity is briefly discussed.

(J. Mandel)

669

MYERS, J. L. (University of Massachusetts, Amherst) **7.3** (9.2)
On the interaction of two scale variables—*In English*
Psychological Bulletin (1959) **56**, 354-391 (3 references, 4 tables)

Statistical interaction has been most commonly defined as a measure of the joint effect of p variables upon performance. If a scaled independent variable is involved, the interaction sum of squares may be further analysed to yield more information than the above definition would indicate. This paper presents a complete analysis of the $(a-1)(b-1)$ degrees of freedom involved in the interaction of two scaled independent variables, A and B. If B is a scaled independent variable, with equal steps, the $(b-1)(a-1)$ degrees of freedom can be analysed into $(b-1)$ components each on $(a-1)$ degrees of freedom. One could compare the slopes of the curves for the quadratic, cubic, or quartic components. If A and B are both scaled variables there are a number of alternative ways of performing this analysis. One could compare the A curves over the levels of B or vice versa.

A proposed analysis that is said to be more efficient stems from one fact: if both variables are scaled, the sum of squares on one degree of freedom can be computed for each of $(a-1)(b-1)$ components of the interaction sums of squares. The mean squares on single degrees of freedom are measures of change in slope and curvature of the B curves over the levels of B.

The author concludes that this procedure extends the present state of comprehension of the interaction term and introduces a method which will permit inferences about the rate of change of slope and curvature coefficients. For this purpose he has developed a rather complicated nomenclature for expressing some of the common methods of handling single degrees of freedom with factorial experiments.

(R. E. Stoltz)

PASTERNACK, B. S. (University of North Carolina, Chapel Hill) **7.6** (9.1)
Analysis of covariance for a 3×4 triple rectangular lattice design (3 associate P.B.I.B.)—*In English*
Biometrics (1960) **16**, 7-18 (16 references, 4 tables)

Covariance is directly dependent on the solution of ordinary problems of analysis of variance for any type of design. This concept was first pointed out by Fisher [*Statistical Methods for Research Workers* (1934). Edinburgh: Oliver & Boyd] and has been emphasised by a number of writers since that data. Hence for any design such as a 3 by 4 triple rectangular lattice the solution is straightforward.

This paper by Pasternack provides an illustrated numerical example for computational purposes of the analysis of covariance for a 3 by 4 triple rectangular lattice and other similar classes of lattice designs (using intra-block analysis). He outlines in detail the procedure and applies it to data from a variety test on corn conducted in Peru in 1955. A discussion is given of the problem of uncertainty as to what effects may be represented by error (" internal ") regression or treatment (" external ") regression.

(B. Harshbarger)

671

RICHTER, D. L. & GEISSER, S. G. (Nat. Inst. Mental Health, Bethseda, Maryland) **7.2** (3.4)
A statistical model for diagnosing zygosis by ridge-count—*In English*
Biometrics (1960) **16**, 110-114 (3 references, 2 tables)

A variance component model is proposed to explain the ridge-counts of siblings in multiple births. Under the assumption that ridge-count is a normally distributed random variable, a test of the hypothesis that a pair of twins is monozygotic is proposed. Estimates of within-egg and between-egg variances have been computed from the twin ridge-count data of Smith & Penrose. The relative probabilities of dizygosis given a ridge-count diagnosis are compared with those of Smith & Penrose [*Ann. Hum. Genet.* (1955) **19**, 273-289] and Lamy *et al.* [*Ann. Hum. Genet.* (1957) **21**, 374-396]. The author also extends the model to triple and quadruple births.

The test statistics may be expressed as weighted sums of chi-square variates under the hypotheses of different combinations of zygosis; their probability density functions may be obtained by differentiating the associated Robbins-Pitman series [*Ann. Math. Statist.* (1949) **20**, 552-560].

(D. F. Morrison)

672

RYAN, T. A. (Cornell University, Ithaca, N.Y.) **7.7** (5.1)

Multiple comparisons in psychological research—*In English*

Psychological Bulletin (1959) **56**, 26-47 (23 references, 1 table)

The logical analysis of the several methods available for comparing group means which have arisen from a single analysis is presented. Special consideration is given to the effects of various methods in regard to the concept of error rate developed by Tukey. Three specific error rates are discussed: error rate per comparison—this is the probability that any particular one of the comparisons will be incorrectly considered to be significant; error rate per experiment—this is the long run average number of erroneous statements per experiment or the expected number of errors per experiment; error rate experimentwise—this is the probability that one or more erroneous conclusions will be drawn in the given experiment.

The author draws the following conclusions:

(i) In general, the same procedure should be used, whether the direction of differences has been predicted in advance or not.

(ii) In general, the experiment should be used as the unit in computing error rates, rather than the individual comparison or test.

(iii) Following Tukey's lead, the error rate should be determined on the basis of that null hypothesis which maximises the rate.

(iv) The error rate per experiment is an upper limit for the error rate experimentwise, and therefore provides a conservative test which can be used when the experimentwise rate cannot be computed.

(v) The choice between the two experiment based error rates is usually one of convenience, since they differ but little numerically in the cases where both procedures are available.

(vi) The relative advantage of confidence limits versus significance tests has not been treated in this discussion, but it is pointed out that the two methods do not lead to parallel conclusions in the case of multiple comparisons.

(R. E. Stoltz)

STAUDE, H. (Institut für Pflanzenzüchtung, Kleinwanzleben) **7.0** (11.0)

Shortening of H. O. Hartley's range method for analysis of block experiments—*In German*

Biom. Zeit. (1959) **1**, 261-275 (4 references, 6 tables)

The computational labour involved in the analysis of variance of block experiments by the range method of Hartley can be markedly reduced following the author's modifications. Hartley's statistic q is factorised into two parts: the first one involving only the sampled data, that is to say the means and ranges, the other one v and r, being the number of treatments and blocks respectively and the range multiplier c. Thus a new test-statistic: $q' = 100$ (range of treatment means)+(sum of block ranges) is obtained. The significant values of q' at the $P = 1$ per cent. and 5 per cent. levels, that is to say $100(q_P + cr\sqrt{r})$ are tabulated for $v = 3(1)9$ and $r = 3(1)8$.

The same procedure is followed in the determination of the upper P percentage fiducial limits D'_P and D''_P, with t based on range for treatment mean differences, when *mutually* comparing differences [" multiple t-test " on $\binom{v}{2}$ differences] or testing against the overall mean (v differences) [Behrens, *Zeit. Acker-u. Pflanzenbau* (1955) **99**, 397-402]. The D_P's are factorised into (sum of block ranges) 100 and a factor a_P and b_P, respectively. The values of a_P and b_P at the levels $P = 5$, 1 and 0·1 per cent. are tabulated for v, r as referred to above. The author demonstrates his short-cut method in each case.

In the second part of the paper, the author compares results obtained from thirty-seven field block trials analysed by the orthodox F- and t-tests and the q'- and D-tests, respectively. The analysis of variance comparisons yielded thirty-five " agreements ", that is to say significance at the same level applied (5 and 1 per cent.) or insignificance; in two cases the F-test gave significance at the lower level. The agreement between the orthodox and the range methods in the case of testing individual means against the general mean or vice-versa—both kinds of tests being considered equally informative as to the issue at hand—appears to be not quite satisfactory.

(R. Wette)

DARROCH, J. N. (University of Manchester) **8.3** (4.3)

The multiple-recapture census. II. Estimation where there is immigration or death—*In English*

Biometrika (1959) **46**, 336-351 (4 references)

The previous paper (I) to which this is a companion [*Biometrika* (1958) **45**, 343-359: abstracted in this present journal, No. 99, **8.3**] referred to the multiple-recapture census. The case discussed was the estimation of a closed population. In this present paper (II) restrictions are removed and the population is subject to augmentation from outside and departure from inside.

The objective of (II), as in (I), is to provide probability models for the observed frequencies of individuals and to give estimates of the parameters of these models. When there is either immigration or death these estimates are obtained, and their variances, and they are shown to be asymptotically efficient. When there is immigration and death operating at the same time the model chosen is unwieldy mathematically. The author is able to obtain estimates of the parameters intervening, but only states how to obtain the variances of these estimates. Neither in (I) or (II) is there any discussion of tests of the basic underlying assumption that at each stage all individuals are equally likely to be captured. The testing of hypothesis about $\{p_i\}$, where p_i is the probability that any member of the population is caught in the ith sample ($i = 1, 2, \ldots, 5$) and about $\{\phi_i\}$, where ϕ_i is the probability of survival at the ith stage, is discussed.

(Florence N. David)

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DERMAN, C., JOHNS, M. V., Jr. & LIEBERMAN, G. J. (Columbia, Stanford and Stanford Universities) **8.8** (8.9)

Continuous sampling procedures without control—*In English*

Ann. Math. Statist. (1959) **30**, 1175-1191 (4 references, 1 figure)

This paper presents two continuous sampling procedures which are modifications of the Dodge CSP-1 procedure [*Ann. Math. Statist.* (1943) **14**, 264-279]. The first denoted CSP-4, is as follows:

(i) Commence complete inspection and continue until i units without defect.

(ii) Stop complete inspection and inspect only $1/k$ items.

(iii) If a sample unit is found defective, resume complete inspection after the end of the sample set of the k items concerned: and so on.

(iv) Correct or replace defective items found.

The second procedure, denoted CSP-5 is the same as CSP-4 except that

(iii)′ If a sample unit is found defective, screen the remaining $k-1$ items in that sample—resume complete inspection, and so on.

It is suggested that these new procedures are useful in practice because of the reluctance to pass a sample of items in which one has been found defective.

The theorems and proof of the expression of the Average Outgoing Quality Limit without the assumption of control for CSP-4 and CSP-5 are given in section three: and under a state of statistical control in section four. The following section develops the original CSP-1 procedure in the direction of the use of probability sampling and without the assumption of statistical control. It is shown that, when the strategy is to produce all defective items during the partial sampling and all non-defective items during complete sampling, the average outgoing quality limit for the CSP-1 with probability sampling is $(k-1)/(k+i)$.

A set of curves is given for determining the values of k and i for a given value of the average outgoing quality level using CSP-4 and CSP-5 procedures under control.

(W. R. Buckland)

GATES, C. R. & FEAREY, J. P. (California Institute of Technology, Pasadena) **8.8** (11.1)
Accuracy requirements for acceptance testing of complex systems—*In English*
J. Amer. Statist. Ass. (1959) **54**, 447-464 (2 references, 15 figures)

The problem considered in this paper involves a procedure for acceptance testing of the constituent components of a complex system with total system performance as the characteristic of interest. It is assumed the individual components are tested by a gauge to determine if some pertinent characteristic lies between two limits with the further assumption that the gauge used is imperfect and that as a consequence these limits become, in reality, two random variables. The additional assumption that the individual component errors are additive and their sum constitute the desired measure of system performance is used.

A mathematical relationship is determined between

(i) the quality of incoming components,

(ii) the characteristics of the gauge, as specified by the accuracy of, and the aperture between, the limits, and

(iii) the quality of the tested system.

Using the above determinations some fifteen figures are presented which graphically depict various relationships between variance of the sum of component errors, test aperture, that is distance between limits, quality of incoming material, standard deviation of "bad" material, and average total fraction rejected. Several similar problems and possible extensions are mentioned.

It is concluded that outgoing quality is sensitive to the aperture of the gauge and insensitive to the accuracy of the limits.

(D. O. Richards)

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GUTHRIE, D., Jr. & JOHNS, M. V., Jr. (Stanford Research Inst. and Stanford Univ., Cal.) **8.8** (1.8)
Bayes acceptance sampling procedures for large lots—*In English*
Ann. Math. Statist. (1959) **30**, 896-925 (12 references)

A lot consisting of N items is characterised by N independent non-negative random variables X_i $(i = 1, 2, \ldots, N)$, with common distribution function $F(x \mid \lambda)$ depending on a single parameter λ, with X_i indicating the quality of the ith item.

A sample of n items is chosen at random from the lot and a design to accept or reject the uninspected remainder of the lot is based on the observed values of the random variables $X_1, \ldots, X_n$. The consequences of this decision are appraised by the following cost model, where $S_k = \sum_{i=j}^{k} X_i$ for $k = 1, 2, \ldots, N$.

Action	Cost
Acceptance	$a_1(S_N - S_n) + a_2(N-n) + s_1 S_n + s_2 n$
Rejection	$r_1(S_N - S_n) + r_2(N-n) + s_1 S_n + s_2 n$

It is assumed that the distribution function $F(x \mid \lambda)$ is a member of an exponential family with $\mathscr{E}(X \mid \lambda) = \lambda$. Furthermore, the values of the parameter λ are assumed to have an *a priori* probability distribution $G(\lambda) = \rho(\Lambda < \lambda)$, where λ is regarded as a random variable Λ. The problem is thus reduced to one of finding the Bayes decision rule for given n and then the Bayes sample size $n^*(N)$.

The essentially unique Bayes decision rule is

$$\delta^*(s_n) = \begin{cases} 1, & \mathscr{E}(\Lambda \mid S_n = s_n) \leqq c, \\ 0, & \text{otherwise} \end{cases}$$

where $c = (r_2 - a_2)/(a_1 - r_1)$, and δ indicates the probability of acceptance of the lot if s_n is the observed value of the sufficient statistic S_n. This rule is shown to be equivalent to

$$\delta^*(s_n) = \begin{cases} 1, & s_n \leqq t(n) \\ 0, & s_n > t(n). \end{cases}$$

The remainder of the paper is devoted to obtaining asymptotic expressions for $t(n)$ as $n \to \infty$, and $n^*(N)$ as $N \to \infty$. The class of *a priori* distributions for which explicit characterisations of $t(n)$ are given includes (i) distributions which are twice continuously differentiable at c with $G'(c) > 0$ and (ii) distributions which assign probability one to a finite number of points. The Bayes sample size $n^*(N)$ is characterised for the above classes of *a priori* distributions $G(\lambda)$ and two classes of distribution functions $F(x \mid \lambda)$, including many commonly arising in practice. It is shown that in case (i) above, $n^*(N)$ is asymptotically proportional to $N^{1/2}$ and in case (ii) above, $n^*(N)$ is asymptotically proportional to $\ln N - \frac{1}{2} \ln \ln N$.

(M. V. Johns, Jr.)

HAMMOND, E. G. (American Cancer Society) **8.5** (8.0)

Inhalation in relation to type and amount of smoking—*In English*

J. Amer. Statist. Ass. (1959) **54**, 35-51 (9 references, 4 tables)

A survey conducted by mail was made to obtain information on inhalation in relation to type and amount of smoking. The proportion of men who said that they inhaled:

(1) increased with amount of smoking and decreased with age,

(2) was very much higher for cigarette smokers than for cigar and pipe smokers, and

(3) was much higher for men who smoked only cigarettes than for men who smoked both cigarettes and cigars.

The proportion of men who said that they inhale differed very little between those smoking filter tip cigarettes and those smoking non-filter tip cigarettes.

A test was made to determine whether the wording of the letter of transmittal enclosed with the questionnaires, the organisation from which the questionnaires were sent, the presence or absence of a postage stamp on the envelope enclosed for reply, or the failure of some men to reply had any influence on the findings. It appeared that these factors made very little difference in the percentage distribution of responses to questions on smoking habits. However, a larger percentage of the addresses replied when a return envelope with a postage stamp attached was enclosed than when a business reply envelope not requiring a postage stamp was enclosed. The wording of the letter of transmittal also seemed to have some influence on the percentage of replies.

(E. G. Hammond)

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MIKULSKI, J., RUDZKI, W. & WIŚNIEWSKI, K. (Math. Inst., Polish Acad. Sci., Warsaw) **8.0** (1.4)

Investigation of the properties of amorphous goods packed in rectangular parallelopiped bales—*In Polish*

Zastowania Mat. (1959) **4**, 332-340.

The main question considered in this paper is the determination of places in a bale of amorphous goods, such as tobacco leaves or cotton, where the value w of the investigated characterististic can be expected to be very close to its mean value w in the whole bale. The authors assume that, in bales which are rectangular parallelopiped in shape the characteristic in question, for example the humidity of tobacco leaves, is constant on surfaces which are homothetical with the surface of the bale, the homothetic centre being the centre of the bale. In other words, the value of w of a characteristic in the point P depends only on the ratio r of the length of segment PS joining P with the centre S of a bale to the length of the segment $P'S$, P' being a point where a half-line directed from S to P intersects the surface of a bale. They check that $w(r)$ is linear then $w(3/4) = \overline{w}$, and if $w(r) = r^2$, then $\overline{w} = w(\sqrt{0{\cdot}6}) = w(0{\cdot}774\ldots)$. From this the authors draw a practical indication as to where a sample from a bale is to be taken.

(S. Zubrzycki)

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KISH, L. & HESS, I. (University of Michigan, Ann Arbor) **8.1** (4.9)
On variances of ratios and their differences in multi-stage samples—*In English*
J. Amer. Statist. Ass. (1959) **54**, 416-446 (8 references, 1 table, 1 figure)

This paper is largely expository in that it aims at providing a comprehensive account of the various complexities of ratio estimators in multi-stage sampling. The authors state their belief that in many instances the techniques of ratio estimators are not readily understood by research workers who conduct and analyse sample surveys. This is particularly important in the computation of variances, where misuse of simple random sampling formulae may result in gross distortion of the true standard error. Numerous and detailed directions are therefore given in specific situations that may arise in sampling practice. Variances of the differences of two ratios are given as they arise in multi-stage sampling. The formulation of these variances is not, to the author's knowledge, explicitly available in the literature.

The paper contains fifteen sections. The first is an introduction and the next five sections give a series of "models" as well as the basic results for the theory of ratio estimators. Later sections of this paper present computationally convenient formulae, with illustrations, and illustrate the models with three actual sample surveys. The last two sections give a more formal outline of the derivations of the variance formulae already presented. The illustrations are carried out largely with numerical data. (Table 427 is misquoted as Table 100 in the text.)

When using the models the authors assume that, either the variance for the ratio $r = y/x$ of two random variables y and x, or of the difference $(r-r')$ of two ratios is desired. It is assumed that for each of the primary selections the sample is self-weighting and the number of primary selections is large so that it is valid to use "combined ratio estimators" and their variance formulae based on Taylor expansions. The primary selections are drawn at random with replacement, and at least two are drawn per stratum: departures from these assumptions are dealt with in subsequent sections. The fourth section discusses sampling without replacement with equal probability as well as other modifications of the basic assumptions. The technique of "collapsed" strata and the case when primary selections are systematically drawn with probability proportional to a measure of size is mentioned. Pseudo-primary selections, that is combinations of several original primary selections within a stratum are discussed, and hints are given as to when these may be useful. The variance of the difference of two ratios is taken up in section five, and expressions to estimate this variance are given in section six. The bias of the ratio estimator is discussed and a rule of thumb given to detect a rather serious bias: if $2 \operatorname{cov}(r-r') > \min[\operatorname{var}(r), \operatorname{var}(r')]$, the bias should be investigated. Formulae for the case of "domains" are also presented.

Numerical illustrations and computationally convenient routines are given with considerable detail in sections seven

KISH, L. & HESS, I. (University of Michigan, Ann Arbor) **8.1** (4.9) *continued*
On variances of ratios and their differences in multi-stage samples—*In English*
J. Amer. Statist. Ass. (1959) **54**, 416-446 (8 references, 1 table, 1 figure)

to nine followed by comments and hints in section ten. The next three sections give the design procedure in three sample surveys carried out at the Survey Research Centre (University of Michigan). Two of the illustrations are from surveys on a national basis and the other gives an example of a systematic sample of a city (Detroit).

In the last two sections mathematical derivations are given of the formulae presented in the paper.

(J. N. de Pascual)

ODERFELD, J. (Math. Inst., Polish Academy of Sciences, Warasw) **8.0** (1.4)
Surfaces of mean humidity—*In Polish*
Zastowania Mat. (1959) **4**, 341-349

The author investigates the equation $w(r) = \bar{w}$, where w is the average value of a particular characteristic in a bale of amorphous goods, in a formally and more general case of an arbitrary convex bale and an arbitrary position of the homothetic centre within the bale. He derives upper and lower bounds for the solution of this equation under the assumption (expressed in slightly different notation) that $r^a \leqq w(r) \leqq r^b$; $1 \leqq b < a$, and thus observes that the solution differs very little from 0·75 for considerably large changes of the parameters a and b.

(S. Zubrzycki)

ROBERTS, S. W. (Bell Telephone Labs., New York City) **8.9** (11.1)
Control chart tests based on geometric moving averages—*In English*
Technometrics (1959) **1**, 239-250 (1 reference, 6 figures)

Of all the points $\bar{X}_i$ ($i = 1, 2, \ldots, j$) on an $\bar{X}$ control chart at time j, only the latest, $\bar{X}_j$, is ordinarily considered in testing the hypothesis that the current process average μ_j equals its nominal value μ_0, which is represented by the central line on the control chart. The effectiveness of the testing procedure can be enhanced in some applications by using statistics which effectively apply a weight pattern to sets of consecutive control chart points. The most familiar is the moving average, $Y_j = (\bar{X}_{j-k+1} + \ldots + \bar{X}_{j-1} + \bar{X}_j)/k$, which gives equal weights to the k most recent points. This paper is primarily concerned with tests based on the " geometric " moving average,

$$Z_j = (1-r)^j \mu_0 + r(1-r)^{j-1}\bar{X}_1 + r(1-r)^{j-2}\bar{X}_2 + \ldots$$
$$+ r(1-r)\bar{X}_{j-1} + r\bar{X}_j = (1-r)Z_{j-1} + r\bar{X}_j, \quad (0 < r \leqq 1),$$

wherein the entire history of $\bar{X}$'s is assigned weights, with weights decreasing as a geometric progression from the latest point back to the first.

If the $\bar{X}$'s are independent and have a common standard deviation, $\sigma_{\bar{X}}$, then limit lines on control charts exhibiting the two types of moving averages would be based on $\sigma_Y = \sigma_{\bar{X}}/\sqrt{k}$ and $\sigma_Z = \sqrt{r/(2-r)}\ \sigma_{\bar{X}}$, respectively.

Based on process simulations in which the process average is allowed to jump from its nominal level μ_0 to $\mu_0 + \delta$, where it remains until detected, it is shown that with respect to early detection of the presence of a changed process average:

(1) for δ greater than about $3\sigma_{\bar{X}}$ the usual test giving the entire weight to $\bar{X}_j (k = 1,\ r = 1)$ is most effective, and for increasingly smaller values of δ increasingly less weight should be given to $\bar{X}_j$ (i.e., increase k, decrease r), and

(2) for $\sigma_Y = \sigma_Z$ (i.e., $k = (2-r)/r$), tests based on the two types of moving average are roughly equivalent.

The points representing the geometric moving averages are easy to generate graphically on an $\bar{X}$ control chart, and this simplicity is of primary importance in many applications. The following procedure may be used:

(1) Mark point $\bar{X}_1$ at abscissa 1.
(2) Mark $Z_0 = \mu_0$ at abscissa $1 - 1/r$.
(3) Draw a straight line connecting Z_0 with $\bar{X}_1$; the point on this line with abscissa $1-(1-r)/r$ is Z_1.

The next point, Z_2, will be at abscissa $2-(1-r)/r$ on the straight line connecting Z_1 with $\bar{X}_2$, which is plotted at abscissa 2. Both the $\bar{X}$'s and the Z's progress one abscissa unit at a time; Z_j is always $(1-r)/r$ abscissa units to the left of $\bar{X}_j$. An introductory value of $r = 2/5$ is recommended, since the Z's will then fall on vertical lines drawn half-way between vertical lines of the $\bar{X}$'s, and also the limit lines for the Z's will be based on $\sigma_Z = \sigma_{\bar{X}}/2$. In some applications preference may be shown to the test that indicates that a change has occurred if either $\bar{X}_j$ or Z_j fall outside their respective limit lines. The properties of some such tests are also considered in the paper.

(S. W. Roberts)

SEAL, K. C. (Calcutta University) **8.8** (8.7)

A single sampling plan for correlated variables with a single-sided specification limit—*In English*

J. Amer. Statist. Ass. (1959) **54**, 248-259 (1 reference)

The author proposes one-sided sampling plans for variables. These plans are of standard structure that utilise

(i) N measurements of the variable of interest y and several auxiliary variables x in an initial inspection lot, and

(ii) n additional measurements on the variables x in the inspection lot of current interest.

The characteristics of the plan and the initial computations depend on the within-lot variance of y and the within-lot multiple correlations between y and the auxiliary variables x. These are assumed known, either from the sampling (at rate N) from the initial lot, or from other sources. Detailed computations are given for

(*a*) the case of a single error rate specification,

(*b*) the case of two error rate specifications, and

(*c*) of the case of minimum cost specifications with given unit inspection costs.

(H. T. David)

SUZUKI, Y. (Inst. Statist. Math., Tokyo) **8.8** (1.8)

On sampling inspection plans—*In English*

Ann. Inst. Statist. Math., Tokyo (1959) **11**, 71-79 (3 references)

The author points out two main defects of Dodge-Romig's sampling inspection plan, which is a typical one of the present sampling inspection plans by attribute, and proposes a different sampling inspection plan based on the Bayes principle. He says that one of the main defects of the Dodge-Romig plan is that the available information about the quality of lots or properties of the manufacturing process is not fully utilised. The author states that while only the cost of sampling and inspection is considered, it is also important to take into account various costs attached to inspection activity and its results.

In this paper, two kinds of losses are introduced: the one due to acceptance of the lot of a given fraction which is defective and the other due to the rejection of it. Further, the *a priori* distribution of the quality of lot is assumed to be known. The inspection plan is similar to Dodge-Romig's single sampling inspection plan, in that it is specified by the size of sample and the acceptance number. For a fixed size of sample, the optimal acceptance number is determined as the one which attains the minimum of the expected loss function with respect to the *a priori* distribution, that is, the Bayes risk: this Bayes acceptance number can be obtained by means of the lemma in this paper. The optimal sample size for use in this plan is decided by minimising the sum of the Bayes risk and sampling cost. As an example, the author treats the case where the *a priori* distribution is of the Beta type.

(M. Siotani)

TINGEY, F. H. & MERRILL, J. A. (Phillips Petroleum Company, Idaho) **8.8** (8.7)
Minimum risk specification limits—*In English*
J. Amer. Statist. Ass. (1959) **54**, 260-274 (1 reference, 1 table)

In determining specification limits for items of product to be subjected to 100 per cent. inspection both process variation and measurement error must be considered. Often costs associated with misclassification are of importance. These costs depend upon the nature of the misclassification and the amount by which an accepted item is defective or a rejected item is acceptable. The costs, with the process and measurement variations, form the basis for defining the risk to be identified with the inspection scheme. Mathematical formulae are derived and tables given which facilitate the construction of minimum risk specification limits. A variety of assumptions about relations among costs and relations between inherent process variation and measurement error are covered.

(F. H. Tingey)

WETHERILL, G. B. (Birkbeck College, London) **8.8** (5.7)
The most economical sequential sampling scheme for inspection by variables—*In English*
J. R. Statist. Soc. B (1959) **21**, 400-408 (8 references, 1 table)

Batches of items are presented for acceptance inspection, and the result of inspection on a single item is represented by a random variable x, with a probability density $\phi(x \mid \theta)$ depending on a parameter θ. A two-point prior distribution is assumed for θ, in which θ takes on two values with probabilities a_1 and $(1-a_1)$. The optimum test procedure is a likelihood-ratio sequential test, and this reduces to a random walk along a line between two absorbing boundaries. Risk functions are assumed for the losses due to wrong decisions, and the equations for the most economical boundaries are formulated involving the probability of acceptance and average sample size at the boundary points. These latter quantities can be determined approximately by a method due to Page [*J. R. Statist. Soc.* B (1954) **16**, 136-139]. Two examples are given, both dealing with a normal probability density; one in which the two points of the prior distribution represent different means but equal variances, and another in which both the means and the variances differ.

The equations for the most economical boundaries are solved by an iterative procedure. The examples demonstrate the speed with which this converges, and also that the approximation involved by using Page's method is negligible.

If sampling is to be done in (constant) groups of size n, then for the normal distribution, the problem of finding optimum boundaries to the likelihood-ratio test reduces to dealing with groups of size one, but with fictitious cost functions. This has an important application to group sequential sampling of items classified " effective " or " defective ", when the group size is large enough to use the normal approximation. A numerical example of this application is given.

(G. B. Wetherill)

ZARKOVIC, S. S. (Food & Agric. Organisation, Rome) **8.2** (8.1)

On the efficiency of sampling with varying probabilities and the selection of units with replacement—*In English*

Metrika (1960) **3**, 53-60 (18 references, 1 table)

In the case of sampling with replacement the efficiency of sampling with varying probabilities (*b*) is compared with the efficiency of some other methods:

(*a*) sampling with equal probabilities
(*c*) ratio method
(*d*) difference method
(*e*) stratified sample with proportional allocation of units
(*f*) regression estimate.

The usual notation is introduced,

x_i = the value of the characteristic for the *i*th unit in the sample
p_i = the probability of the selection of the *i*th unit
y_i = the measure of size of the *i*th unit

$$(p_i = y_i/y,\ \Sigma\, y_i = y).$$

In the first part of the paper a single stage design is considered. For the purpose of comparison the expected value of the variance for the estimate of the mean is expanded in Taylor series. Terms with powers higher than second order are neglected. The result depends mainly on the relative covariance between x^2 and p and on the variance of p. A comparison of the efficiencies of (*a*) and (*b*) is now possible.

If a correlation is assumed between the value of the characteristic x and the measure defining the selection probabilities y, the efficiency of (*b*) depends on the correlation coefficient of x and y. The alternative methods (*c*), (*d*), (*e*), (*f*) are considered. Their variances are compared in table 1 together with those of methods (*a*) and (*b*). Analogous comparisons are possible by using the same method of expanding the variance formulae for a mean in Taylor series in complex designs. This is shown in the second part by comparing the efficiency of combinations of (*a*) and (*b*) in a two-stage design.

(W. Piesch)

BOX, G. E. P. & DRAPER, N. R. (Princeton and North Carolina & I.C.I., Welwyn, England) **9.3** (9.4)

A basis for the selection of a response surface design—*In English*

J. Amer. Statist. Ass. (1959) **54**, 622-654 (19 references, 2 tables)

This paper discusses certain requirements for a response surface design and studies the properties of a design satisfying the requirements.

The experimenter is fitting a function $f(\xi) = f(\xi_1, \xi_2, \ldots, \xi_k)$ over some "region of interest" R in ξ-space and if the true function for the response is $y = g(\xi)$; then in order that the fitted function gives a good representation of the response surface, a design should have minimum value for the expected mean-squared-error averaged over the region defined by

$$\mathcal{J} = \frac{N}{\sigma^2} \int_R \mathscr{E}[\hat{f}(\xi) - g(\xi)]^2 d\xi \bigg/ \int_R d\xi,$$

where σ^2 is the per-observation variance and N is the total number of observations. This expression for $\mathcal{J}$ can be split up into two parts: $\mathcal{J} = V+B$, where V is N/σ^2 times the average variance of the estimate $\hat{f}(\xi)$ of $f(\xi)$ and B is N/σ^2 times the average squared bias resulting from using $f(\xi)$ as an approximation to the true function $g(\xi)$.

If the function $f(\xi)$ is a polynomial of order d_1 and $g(\xi)$ is a polynomial of order $d_2 (d_2 > d_1)$, then it is proved that the squared bias B is minimised for all values of the coefficients of the neglected terms, by making the moments of the design up to order $d_1 + d_2$ equal to the corresponding moments of a uniform distribution over the region R.

If the true function is quadratic and a linear model is fitted over a spherical region R defined by $\Sigma\, \xi_i^2 \leqq 1$; then the best design is proved to be a first order orthogonal design of type B. A table is given for the optimum size of the design for different values for the ratio of variance to bias contribution. From the results in the table, the authors conclude that in cases likely to be met in practice the size or the optimal orthogonal design of type B should usually be slightly greater than the one for which the bias alone is minimised, and variance V is neglected.

Designs which satisfy the above essential requirement should also allow a check to be made on the representational accuracy of the assumed model. As a measure for this requirement, the authors take the power of the test for goodness-of-fit, and discuss the designs for which this requirement is also satisfied.

(B. V. Shah)

BUDNE, T. A. (Great Neck, New York) **9.7** (9.2)

The application of random balance designs—*In English*

Technometrics (1959) **1**, 139-155 (15 figures)

Experience has shown that random balance experiments can be a very effective means for screening large numbers of variables in a limited number of samples to find the few which are largest contributors to the effects under consideration. In manufacturing, testing, and development areas, in particular, it is not at all unusual to find non-statistical technical personnel searching for the causes to large effects, among large numbers of variables. It is important that a relatively simple, effective, and economically feasible screening technique be made available to such personnel to use without the assistance of an expert statistician. The author states that random balance design is an answer to this problem. Since experience in industrial situations consistently shows that only a very few out of any large number of suspect variables are major contributors to a problem, the requirements for highly sensitive statistical designs are small and the random balance designs are particularly attractive.

The most useful random balance designs are restricted random samples of full factorial designs. Full factorial and fractional factorial designs may be parts of the total design. A synthesised example illustrates such a design and the techniques of analysis using "scatter plots" for each variable; some non-parametric and other tests of significance are also given.

Actual case histories in industrial situations are described. Where large effects of a variable can be determined with very few tests, the sequential nature of the analysis of test permits the isolation of such a variable and the termination or modification of the experiment at that point.

(T. A. Budne)

DEBAUN, R. M. (American Cyanamid Co., Stamford, Connecticut) **9.3** (9.4)

Response surface designs for three factors at three levels—*In English*

Technometrics (1959) **1**, 1-8 (8 references, 3 tables, 1 figure)

Various 3-level 3-factor experimental designs are compared with respect to their utility in fitting a second degree polynomial. The designs are compared on the one hand to the 3^3 factorial experiment (F) and on the other hand to the five level rotatable central composite designs of Box and Hunter (R). The main designs considered are

(i) the cube and octahedron

(ii) the cube and two octahedra

(iii) the cube octahedron

(iv) the cube and cuboctahedron and

(v) the cuboctahedron and octahedron.

each with varying degrees of replication in the centre point.

As regards approach to rotatability, (iii) and (v) appear to be the most desirable experimental patterns, (i), (ii) and (iv) seeming even less " rotatable " than F.

As regards experimental efficiency (defined as reciprocal variance of the predicted values adjusted for the number of observations in each pattern) almost all the patterns were superior to F and equal to or better than R.

With respect to the aliasing of first order terms by third order terms, none of the designs is as free of possible bias as F while almost all the designs are better than R in this respect.

(R. M. Debaun)

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DRAPER, N. R. (University of North Carolina, Chapel Hill) **9.3** (9.2)

Second order rotatable designs in four or more dimensions—*In English*

Ann. Math. Statist. (1960) **31**, 23-33 (3 references, 2 tables)

This paper is devoted to the construction of second order rotatable designs in four or more dimensions. The method of construction is the same as previously employed by Bose & Draper in their construction of second order rotatable designs in three dimensions [see *Ann. Math. Statist.* (1959) **30**, 1097-1112; abstracted in this journal, No. 495, **9.3**]. The author obtains certain infinite classes of second order rotatable designs in dimensions higher than three by a suitable generation and combination of basic point sets. Six infinite classes of rotatable designs analogous to those formed in three dimensions are tabulated and restrictions are placed upon the levels of the factors in order to keep the number of design points down to a reasonable group. For the designs presented, the number of design points range from 32 to 48 for the case of four factors, from 52 to 100 for five factors, from 88 to 216 for six factors and from 156 to 476 for seven factors. Since the parameters to be estimated in these cases number 15, 21, 28 and 36 respectively, the relative number of design points for some of these designs is quite large.

A method is presented for enlarging a second order rotatable design in $(k-1)$ dimensions to a second order rotatable design in k dimensions. If the design in $(k-1)$ dimensions has $N'+n_0$ points (where n_0 represents the number of centre points), then the design in k dimensions has $N = 2N'+4+n_0$ points. The method for such an extension depends on a relationship among the first four moments, providing that N and N' are satisfied. The method is exemplified by the construction of a 32 point design in 4 dimensions with no centre points from the 14 point cube plus octahedron arrangement in three dimensions without centre points.

(J. W. Wilkinson)

DYKSTRA, O., Jr. (General Foods Corp. Research Centre, Tarrytown, N.Y.) 9.2 (7.1)
Partial duplication of factorial experiments—*In English*
Technometrics (1959) **1**, 63-75 (6 references, 3 tables)

Two-level factorial and fractional factorial designs are described in which a sub-set of the treatment combinations are duplicated. The duplicate runs provide an unbiased estimate of the experimental error variance and more reliable estimates of the effects. If the treatment combinations which are duplicated are selected as a fraction of the basic design, then the analysis remains relatively simple.

Several analysis procedures are described, and a numerical example is given. Designs are proposed for partially duplicated two-level factorial and fractional factorial designs for as many as eleven factors. The designs for six or more factors provide for sixteen duplicate treatment combinations.

(O. Dykstra, Jr.)

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GARDINER, D. A., GRANDAGE, A. H. E. & HADER, R. J. (Oak Ridge Nat. Lab. and North Carolina State Coll., Raleigh) 9.3 (9.7)
Third order rotatable designs for exploring response surfaces—*In English*
Ann. Math. Statist. (1959) **30**, 1082-1096 (4 references, 4 tables)

This paper is concerned with extending the criterion of rotatability, as advanced by Box & Hunter, to experimental designs for estimating response surfaces by third order polynomial equations. This criterion is that the variances of estimates of the response made from the least-squares estimates of the polynomial equations are constants on either circles, spheres or hyper-spheres about the centre of the design. The method of attack has been to examine combinations of regular and semi-regular geometrical figures and find those combinations whose co-ordinate points satisfy the moment properties, to order six, of spherical distributions. Designs with these properties and the attendant restrictions were shown by Box & Hunter to have spherical variance contours when the polynomial coefficients were estimated by the method of least-squares.

It was found that third order rotatable designs in two factors could be attained by locating seven or more experimental points equally spaced on each of two concentric circles of different non-zero radii. It was also shown that certain rotatable designs in two factors can be performed in two stages, so that second order polynomial coefficients can be estimated after the first stage and third order polynomial coefficients after the second stage. By choosing the radii of the two circles in the proper ratio it is possible to obtain estimates of the polynomial coefficients which are independent of " block " effects due to running the experiments in two stages. Such designs were termed sequential third order rotatable designs.

In three factors, designs such as these were presented which consisted of composites of cubes, truncated cubes, octahedra, cuboctahedra, icosahedra and dodecahedra. Two of these designs in three factors were constructed so that they might be performed sequentially.

One sequential third order rotatable design in four factors was also presented: this design has as its experimental points the vertices of the 4-dimensional analogues of a cube, a truncated cube and two octahedra of different dimensions.

(D. A. Gardiner)

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FINNEY, D. J. (University of Aberdeen) **9.4** (9.0)

The economic efficiency of experimentation—*In English*

Publ. Math. Inst. Hung. Acad. Sci. (1959) **4**, 203-224 (21 references, 2 tables)

The responsibility of the statistician consists not only in the analysis of experimental results but also in the design of the experiments. Recognition is now needed that design is only one aspect of the broad planning of research projects, and that the statistician's participation should not be limited to the narrow combinatorial problems of design. Close collaboration between statistician and experimental scientist is essential if the efforts of the latter are to be most effective.

Aspects of both the internal and the external economy of research are considered, the planning of experiments so that the resources of time, materials, or labour available are used efficiently and the determination of the scale of resources that ought to be available in order that the research shall best meet the requirements of the community. For fundamental research, often only the internal economy can usefully be discussed, but for much technological research external factors should also be taken into account. Some of the particular questions that need discussion are listed.

Theoretical and practical discussions relating to special problems of planning: the first one is the choice of the number of factors to be used in factorial experiments; the advantages of including as many factors as the size of the experiment will permit are expressed qualitatively and quantitatively. The second one concerns the use of concomitant information on plots or other experimental units, and the relative merits of using this in the formation of homogeneous blocks or adjusting the results by a covariance analysis on the concomitant.

Deciding the amount of experimentation to be undertaken on a particular issue is discussed by the author and one aspect of this is to decide the optimal number of similar experiments when the aim is to estimate the best level of a particular factor (e.g. amount of fertiliser per unit area under crop) for practical use. This is mathematically much simpler than the determination of the optimal amount of experimentation on which to base a practical decision between two alternative and sharply contrasted lines of action (e.g. the choice between two alternative diets). Analysis of this second form of problem is also discussed for the situation in which some preliminary information exists and the experimenter must either make an immediate decision on the basis of this or specify a further amount of experimentation.

Examples of optimal planning of successive stages of experimentation are given:

(i) Selection of improved varieties of a crop plant by making field experiments in successive seasons and in each season selecting a proportion of the best for continuation in the next season. If the number of initial varieties, the number to be finally selected from the last season, and the total experimental

FINNEY, D. J. (University of Aberdeen) **9.4** (9.0) *continued*

The economic efficiency of experimentation—*In English*

Pub. Math. Inst. Hung. Acad. Sci. (1959) **4**, 203-224 (21 references, 2 tables)

area are fixed, the best allocation of selection fractions and areas to the separate stages is needed; apparently equal fractions in each year and equal division of the whole area constitute a scheme that is usually near to the optimal.

(ii) " Screening " large numbers of chemical compounds in a search for any that have therapeutic value. The initial population contains possibly a few active compounds and certainly a large number of useless ones. A standard test procedure, involving laboratory animals, may be used to discriminate between positives and negatives, and successive stages may be employed exactly as in the varietal selection example. The aim now is that compounds finally selected shall include the highest possible proportion of active ones, subject to restrictions on the costs of the selection programme that will primarily consist in a specification of the total number of animals to be used.

(D. J. Finney)

GLENN, W. A. & DAVID, H. A. (Virginia Polytechnic Inst., Blacksburg) **9.5** (4.3)

Ties in paired-comparison experiments using a modified Thurstone-Mosteller model—*In English*

Biometrics (1960) **16**, 86-109 (11 references, 2 tables, 1 figure)

In making paired comparisons a judge is frequently unable to express any real preference in a number of the pairs he judges. Nevertheless, some of the methods in current use do not permit the judge to declare a tie. In other cases ties are permitted, but are ignored in performing the analysis. Alternatively, the ties are sometimes divided, equally or randomly between the tied members of a pair. From a review of the relevant literature it appears that there is a need, at least in the estimation of response-scale values, for a method which takes tied observations into account.

In the Thurstone-Mosteller method the standardised distribution of the difference of two stimulus responses is normal with unit variance and mean equal to the difference of the two mean stimulus responses. In prohibiting ties the assumption is in effect made that all differences, however small, are perceptible to the judge. In this paper the assumption is made that a tie will occur whenever the difference between the judge's responses to the two stimuli lies below a certain threshold, i.e. if the difference lies between $-\tau$ and τ the judge will declare a tie. The parameter τ and the mean stimulus responses are estimated by least squares.

To overcome a difficulty presented by correlated data, an angular reponse law is postulated for the response-scale differences. In the resulting transformed data heterogeneity of variances is encountered. A weighted solution is set up, the weights being determined from a preliminary unweighted analysis. This results in an iterative procedure. Large-sample variances and covariances of the estimates are obtained. A test of the validity of the model is described. A computational procedure is given and exemplified through application to experimental data.

It is shown that the proposed method is applicable to non-balanced experiments. These include cases in which unequal numbers of observations are made on the various pairs, as well as cases in which no observations are made on certain of the pairs.

(W. A. Glenn)

MORAN, P. A. P. (Australian National University, Canberra) **9.7** (9.4)

The power of a cross-over test for the artificial stimulation of rain—*In English*

Aust. J. Statist. (1959) **1**, 47-52 (2 references)

A particular design (A) for testing the artificial stimulation of rain is to measure the precipitations in a target area and a control area during $m+n$ rainfall periods, the target area having been seeded during m of these periods selected at random, and the control area unseeded for all the $m+n$ periods. A variant (B) is to seed the control area during the n periods in which the target area is unseeded. Although non-parametric tests are valid with fewer assumptions, more powerful tests based on analysis of variance and covariance are desirable in order to reduce costs.

The author considers tests based on the following assumptions: x and y, the logarithms (or other normalising transforms) of the precipitations in the target and control areas, have a bivariate normal distribution, whose variances V_1, V_2 and covariance C are not affected by seeding of either area; seeding of either area increases the mean for that area by some quantity D which is the same for both areas. Two methods of analysing the data are

(1) to test the difference between the means of the seeded and unseeded x's after correction by the covariate y, and

(2) to test the difference between the means of $x-y$ for the target seeded and unseeded periods, after correction by the covariate $x+y$.

Method (2), being symmetrical, is appropriate for design (B).

The ratio of the asymptotic (m and n both large) power of design (B) with method of analysis (2) to that of (B) with analysis (1) is shown to be $V_2(V_1+V_2+2C)(V_2+C)^{-2}$. The ratio of the power of (B) with method of analysis (2) to that of (A) with analysis (1) is $(V_1+V_2+2C)V_2^{-1}$. If $V_1 = V_2$ and $C = 0$, the latter ratio is 2; if $V_1 = V_2$ and the correlation is high, the ratio will be near four. These results indicate that, with appropriate analysis, double seeding usually leads to a great increase in power.

(E. W. Bowen)

OGAWA, J. (University of North Carolina, Chapel Hill) **9.1** (0.6)

A necessary condition for existence of regular and symmetrical experimental design of triangular type, with partially balanced incomplete blocks—*In English*

Ann. Math. Statist. (1959) **30**, 1063-1071 (10 references)

In this paper, the author discusses a scheme of triangular association in which there are $v = n(n-1)/2$ treatments. This particular scheme was defined in a paper by Shrikhande [*Ann. Math. Statist.* (1959) **30**, 39-47; abstracted in this journal, No. 292, **9.1**] The design is called a partially balanced incomplete block design of the triangular type if these v treatments are applied to b blocks, each having k experimental units, in such a way that

(i) each block contains k different treatments,
(ii) each treatment occurs in r blocks,
(iii) any two treatments occur together in λ_i blocks if they are ith associates ($i = 1, 2$).

The paper presents the conditions necessary for the existence of certain designs of the above type with $v = b$ and $r = k$ in terms of the Hasse-Minkowski p-invariant. For a discussion of the latter, see Jones [*The Arithmetic Theory of Quadratic Forms* (1950). New York: Wiley] and another paper by Shrikhande [*Ann. Math. Statist.* (1950) **21**, 106-111].

Examples of non-existent partially balanced incomplete block designs of the triangular type are given for $n = 7$:

(i) $r = k = 6$; $\lambda_1 = 0$, $\lambda_2 = 3$;
(ii) $r = k = 10$; $\lambda_1 + \lambda_2 = 9$; $\lambda_1 = 0, 1, 2, 3, 8$.

(S. S. Shrikhande)

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OLDHAM, P. D. (Pneumoconiosis Research Unit, Penarth, Utd. Kingdom) **9.0** (9.9)

Clinical and field trials—*In English*

Acta Medica. Acad. Sci. Hung. (1959) **14**, 313-326 (6 references, 2 tables)

The author discusses methods useful for precisely defining the selection of subjects for trials and the criteria of success are examined.

Designs for eliminating bias are treated: for example, the blind and double blind trials, dummies, placebos, separating the treatment itself from its assessment, after-effects and Wilson squares. Their special merits and limitations are investigated and methods for large-scale trials are contrasted with those appropriate to small-scale experiments. The different aspects for summarising the results are also discussed.

(I. Juvancz)

702

SCHEFFÉ, H. (University of California, Berkeley) **9.7** (9.3)
Experiments with mixtures—*In English*
J. R. Statist. Soc. **B** (1959) **21**, 238

The author notifies a correction to formulae (4.8) on page 349 of his paper already abstracted in this journal No. 116, **9.7**.

(W. R. Buckland)

SHAH, B. V. (University of Bombay) **9.1** (0.6)
A generalisation of partially balanced incomplete block designs—*In English*
Ann. Math. Statist. (1959) **30**, 1041-1050 (8 references, 2 tables)

Partially balanced incomplete block designs have been used in experimental situations for over two decades. They were first defined by Bose & Nair [*Sankhyā* (1939) **4**, 337-372], and have been found to include most of the designs in practical use. By relaxing certain conditions in the definition of such designs, the author has provided a new class of designs which has the property that the analysis of the new designs is on almost the same basis as that of the partially balanced incomplete block designs.

The theory is based on certain properties of the reduced normal equations for the intra-block estimates of treatment contrasts ($\hat{t}_i$) based on the usual linear model for the yield of the ith-treatment in the jth block ($i = 1, 2, \ldots, v$; $j = 1, 2, \ldots, b$): ($y_{ij} = m+b_j+t_i+e_{ij}$). Each block contains k experimental units, each having a different treatment ($k \leqq v$), and each treatment is applied to r experimental units ($r \leqq b$). These equations are $\mathbf{Q} = \mathbf{C}\hat{t}$, where $\mathbf{Q}$ is the vector of adjusted treatment totals, and $\mathbf{C}$ is the corresponding matrix of adjusted sums of squares and products. If the design is connected, i.e., every treatment contrast can be estimated, $\hat{t} = [\mathbf{C}+a\mathbf{E}(v, v)]^{-1}\mathbf{Q}$, where a is any non-zero real number and $\mathbf{E}$ is a matrix with all elements unity. Consider the $v(v+1)/2$ treatment pairs, including a treatment with itself. These are partitioned into $m+1$ disjoint classes, which can be called the association classes; define the corresponding association matrices ($\mathbf{B}_h$) with elements 1 or 0 depending on whether two treatments belong to a given association class or not. $\mathbf{C}$ can be written as a simple linear function of the $\mathbf{B}_h$. If for all t and x, it can be shown that $\mathbf{B}_t\mathbf{B}_x+\mathbf{B}_x\mathbf{B}_t$ is a certain linear function of the $\mathbf{B}_h$ with the coefficients simple functions of the parameters of the design, the analysis of the design will be identical with that of a partially balanced incomplete block design. An example is given for a special design with $v = 6$, $b = 9$, $r = 3$, $k = 2$, $m = 4$.

(R. L. Anderson)

SHIRAFUJI, MICHIE. (Kyushu University) **9.3** (9.7)

A two-stage sequential design in response surface analysis—*In English*

Bull. Math. Statist. (1959) **8**, 115-126 (8 references, 4 tables)

In analysis of a k-dimensional response surface of the second degree, the author first makes the 2^k type experiment, and tests the effect of interaction ($x_i x_j$) terms. If this effect is significantly large, the author takes as an approximation model a surface without square (x_i^2) terms and estimate the coefficients of other terms. If the effect is not large, the author makes additional $2k+1$ observations according to the Box-Wilson composite-type design with total 2^k+2k+1 observations, and estimate the coefficients including square terms.

To compare the estimates in the two-stage experiment with ones in the single-stage 2^k type and composite type experiments, the norms defined as the volume integral over k-dimensional sphere of the mean-square error of the estimated surface are computed. Some numerical results are also tabulated.

(M. Sibuya)

VARTAK, M. N. (University of Bombay) **9.1** (0.6)

The non-existence of certain partially balanced to incomplete block designs—*In English*

Ann. Math. Statist. (1959) **30**, 1051-1062 (6 references)

The author considers partially balanced incomplete block designs with three associate classes for v_1v_2 treatments having a rectangular association scheme, which is defined as follows: the v_1v_2 treatments are arranged in the form of a $v_1 \times v_2$ rectangle. Any two treatments in the same row are first associates, any two treatments in the same column are second associates and any two treatments not in the same row and not in the same column are third associates. Each of the v treatments is allocated to r blocks, each block to contain k different treatments, such that any two treatments of the ith associate class occur together in λ_i blocks ($i = 1, 2, 3$). The parameters r, k and λ_i are related to v_1 and v_2 by this equation

$$r(k-1) = (v_2-1)\lambda_1+(v_1-1)\lambda_2+(v_1-1)(v_2-1)\lambda_3.$$

There is a misprint of this in the article, equation (1.7). For definitions of the terms used here see the paper by Bose & Shimamoto, " Classification and analysis of partially balanced incomplete block designs with two associate classes " [*J. Amer. Statist. Ass.* (1952) **47**, 151-184].

The author derives a number of necessary conditions for the existence of such designs, and in particular makes use of Hasse-Minkowski invariants to prove the impossibility of a number of such designs. Special theorems are proven with regard to symmetrical designs of this type, that is where $v = b$.

The following examples of non-existent symmetrical designs are given:

$v = b$	$r = k$	λ_1	λ_2	λ_3
24	8	4	7	1
66	14	7	4	2
48	10	5	4	1
87	16	4	8	2
63	11	4	5	1

Two non-symmetrical partially balanced incomplete block designs of this type are also shown to be impossible:

v	b	r	k	λ_1	λ_2	λ_3
30	20	10	15	10	8	33
30	50	10	6	5	6	0

(S. S. Shrikhande)

ZELEN, M. (National Bureau of Standards, Washington, D.C.) **9.2** (2.5)
Factorial experiments in life testing—*In English*
Technometrics (1959) **1**, 269-288 (16 references, 10 tables)

This paper discusses procedures for analysing factorial experiments, where the experiment deals with the life testing of components or equipment. These procedures assume an underlying general distribution of "times-to-failure", of which the exponential, Weibull, and extreme value distributions are special cases. Statistical tests and confidence procedures are outlined, and an example illustrating the procedure for life-test results of glass capacitors is included. Small sample approximations, which are adequate for practical applications, are given for the proposed procedures. This is shown empirically by generating thousands of life-test experiments on an electronic computer. An empirical sampling investigation is given of the robustness of the proposed procedures.

From the sampling results it is concluded that these techniques are sensitive (non-robust) to departures from the original assumptions on the probability distribution of failure-times. An investigation is also given of a transformation which appears to give robust results. These same techniques carry over exactly to the situation where one is analysing an array of variance estimates from an underlying normal population.

The experimental situation considered is an a times b factorial experiment where n "components" are put on life-test for each factorial combination. The test is terminated for a particular treatment combination after the rth failure ($r \leq n$). When the underlying distribution of failure times is exponential, the model taken for the mean time to failure is

$$\theta_{ij} = ma_ib_jc_{ij}$$

where (i, j) refers to a particular treatment combination and the parameters m, a_i, b_j, c_{ij} are analogous to the mean, main effects, and interaction terms of the two-way analysis of variance under Model I.

A revised version of Table 4c is given in a later issue of the journal [*Technometrics* (1960) **2**, 121].

(M. Zelen)

ANDERSEN, F. (Statens Skadedyrlab., Springforbi, Denmark) **10.9** (11.0)

Competition in populations consisting of one age group—*In English*

Biometrics (1960) **16**, 19-27 (18 references)

The author presents a number of mathematical models involving competition in populations consisting of a single age group. Many actual examples relating to these models are cited from the extensive literature on the subject. Some frequently used models follow.

1. The fraction dying (or surviving) during the pre-imaginal development is postulated as being linearly related to the initial number of larvae.
2. A population containing two genotypes may show differential mortality. With appropriate assumptions the surviving fractions of the two genotypes may be postulated as equal to second power polynomials in the initial total.
3. If, in the case of larval cannibalism, it is assumed that (*a*) if two larvae meet, there is a constant probability that one of them die, and (*b*) the probability of one larva meeting one of the other larvae is proportional to their number, then the logarithm of the surviving fraction is linearly related to the initial number of larvae.
4. The number of eggs per female is postulated as being linearly related to the reciprocal of the number of animals per unit of food which is in turn proportional to the amount of food or the number of oviposition sites per animal.
5. The effect of mutual interference with oviposition is postulated to add a linear term to the model described in 4 above.
6. Several models are suggested involving the sex ratio, one being a linear relation between the proportion of females and the initial number of larvae per unit of food arising from differential mortality.
7. Some mention is made of relationships between rate of development and density but no specific mathematical model is presented.
8. Similarly comments are made on a relationship between movements and density.
9. An example is cited in which rate of development and mortality decrease with increasing density.

The author points out that populations consisting of one age group are not at all uncommon. Many insects emerge and lay their eggs almost on the same day, so that the next generation consists of larvae of practically equal age. Similar populations are encountered among some fish and some birds.

Experiments are suggested to discover the limitations of the models mentioned and to note which deviations from the models occur. Further experiments are proposed to suggest ways of relating the distribution in time of the effect of density within a single age group.

(R. J. Taylor)

709

ATKINSON, R. C. (University of California, Los Angeles) **10.1** (0.6)

A Markoff model for discrimination learning—*In English*

Psychometrika (1958) **23**, 309-322 (16 references, 1 table, 3 figures)

This paper is a preliminary attempt to develop a quantitative theory of discrimination learning which incorporates the concept of an orienting response. The theory is developed in detail for experimental procedures in which two stimuli are employed and two responses are available to the subject, but the formulation can be extended readily to certain n-response situations. Applications of the model to cases involving probabilistic and non-probabilistic schedules of reinforcement are considered and some predictions are derived and compared with experimental results.

It is assumed that the orienting and non-orienting responses are elicited by a set of stimuli associated with the beginning of the trial. The conditioning process is elaborated and the probability of a response in the presence of particular stimulus elements is defined as the proportion of stimulus elements conditioned to the response.

Using four possible stimulus elements, and assuming that all are conditioned to the same response at the start of the first trial, an organism can be described as being in one of sixteen possible states on any trial. For these conditioning assumptions and the experimental parameters of β, the probability of occurrence of stimulus T_1; π_1, the probability that response A_1 following T_1 is correct and π_2 the probability that an A_1 response following a T_2 is correct, a transition matrix **P** describing the learning process can be derived. Let $u_i(n)$ be the expected probability of being in state i at the start of trial n, where the first experimental trial is $n = 0$. Define the row matrix

$$\mathbf{U(n)} = [u_1(n), u_2(n), \ldots, u_{16}(n)].$$

Then **P** represents the one-stage transition matrix of order sixteen where p_{ij} is the conditional probability of being in state j on trial $n = 1$, given that the system was in state i on trial n. The Markoff process describing discrimination learning at the start of trial n is

$$\mathbf{U(n)} = \mathbf{U}(0)P^n.$$

The assumptions for conditioning and response probability are different from those postulated by Estes & Burke in their stimulus sampling model, but the ideas fundamental to the model presented in this paper were formalised initially within the framework of the Estes & Burke stimulus sampling theory.

To illustrate certain aspects of the author's theory, a study of Estes & Burke and one of Popper & Atkinson are described. He concludes that qualitatively it appears that the theory accounts for some aspects of traditional types of discrimination learning and can be extended without modification to discrimination problems involving probabilistic reinforcement schedules.

For this model, predicted asymptotes always lie between the predictions of Buch-Mosteller and Burke-Estes. He believes the model generates some interesting predictions regarding both reinforcement schedules and similarity between discriminanda. He cites some possible objections that might be raised, but believes the resulting complications are not unmanageable and that if modification proves necessary, theoretical predictions still can be generated easily. (R. E. Stoltz)

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AUDLEY, R. J. (University College, London) **10.1** (2.5)

A stochastic model for individual choice behaviour—*In English*

Psychological Review (1960) **67**, 1-15 (18 references, 1 table)

A stochastic model is presented which is concerned with the inter-relations of response variables observed in choice situations. The model, based on an exponential probability model, presented by the author is based on two assumptions:

(i) It is assumed that for given conditions of stimulus and organism, there is associated with each possible choice response a single parameter. This parameter determines the probability that in a small interval of time $[t, t = \Delta t]$, there will occur an implicit response at a time with which the parameter is associated.

(ii) It is assumed that a final choice response is made when a run of K implicit responses of a given kind appears, this run being uninterrupted by occurrences of implicit responses of other kinds.

The theory outlined in this paper is limited to situations involving the choice between only two alternatives, that is $m = 2$ in the case where $K = 2$. Applications of the model to empirical data on psychophysical discriminations, relation of judgment time to perceived distance between stimuli, relationship between confidence decision time and perceived distance between stimuli in preference and conflict situations are presented. The author discusses the extension of the model to cases arising when K is equal to more than 2.

(R. E. Stoltz)

711

BASHARIN, G. P. **10.1** (2.5)

Final probabilities for multi-dimensional Markoff processes which describe the action of some two-stage telephone bi-signal systems—*In Russian*

Teor. Veroyat. Primen. (1958) **3**, 452-458 (7 references)

Final probabilities are determined for multi-dimensional Markoff processes with continuous time and a finite number of states describing the action of a two-stage telephone system with one switch in the second stage.

It is assumed that service-calls form independent Poisson streams of calls, and that the service times have independent negative exponential distributions.

On the basis of these final probabilities, some other probability formulae for a common number of the busy lines are determined. These formulae are extensions of the well-known Erlang's formulae in the multi-dimensional case. Common group selection and random occupation of each free connecting device are considered.

(G. Basharin)

712

BRUCKMANN, G. & PFANZAGL, J. (Universität Wien) **10.0** (0.2)

Literature survey on the decomposition of time series influenced by seasonal variations—*In German*

Blä. Dtsch. Ges. Versich.-math. (1960) **4**, 301-309 (57 references)

This review paper deals comprehensively with the existing literature on the decomposition of time series, the course of which is influenced by several factors. Decomposition aims at the elimination of certain influences which some of these factors have on a given series.

The first part of the article deals with the general problems of decomposition of time series. Methods are basically distinguished as either " extrinsic " methods or " intrinsic " ones, as proposed by Wald. " Extrinsic " methods are based on models which separately contain the influencing factors under consideration and which describe their influence on the given series quantitatively. An application of these " extrinsic " methods requires the use of additional statistical information. " Intrinsic " methods aim at obtaining the quantitative influence of the relevant factors from the given data alone. Hence, it is essential that these influences show a different time behaviour, in order to be separable.

Usually, time series are decomposed into three components:

(i) A " smooth " component, reflecting the influence of steadily and slowly varying factors (trend plus cyclical movements);

(ii) The " seasonal " component, reflecting seasonal variation;

(iii) The " irregular " component, comprising any short-run or other irregular influences on the series.

Models that permit adequate consideration of the influence of random variations on the future course of a series lead to stochastic processes; however, the theory of stochastic processes as applied to the analysis of economic time series has not yielded satisfactory results, notwithstanding its successful applications in other fields. The same methods as used in the decomposition of time series are, furthermore, applicable to an analysis of industrial and agricultural experiments.

A second (special) part of the paper starts with a discussion of the " intrinsic " methods. They are grouped into methods of a " stable seasonal normal " and of a " variable " seasonal movement. With a " stable " seasonal normal the seasonal component is assumed to remain the same year by year; with a variable seasonal movement variations of the form and/or amplitude of the seasonal component from year to year are taken into account. Within the frame of the " intrinsic " methods regression-line technique, the variate-difference-method and harmonic analysis are discussed. Finally, there are paragraphs on " extrinsic " methods, season-corridors and on the use of electronic computers.

(O. Anderson, Jr.)

713

BURKE, C. J. & ROSENBLATT, M. (Indiana University, Bloomington) **10.1** (–.–)

A Markoffian function in a Markoff chain—*In English*

Ann. Math. Statist. (1958) **29**, 1112-1122 (5 references)

This authors pose the following question: if an experimental situation does not permit observation of a Markoff process $X(n)$, but of a derived process $Y(n) = f\{X(n)\}$, is the process $Y(n)$ also Markoffian? Four theorems are given with three corollaries. Part of the interest in the proof of the first and fourth theorems lies in the fact that if the collapsed processes in these cases satisfy the Chapman-Kolmogoroff equation, then they are Markoffian. The importance of these collapsed processes lies in the fact that they form an essential part of the development used by the author: the states i of the original process $X(n)$ are collapsed into a single state of the observed process $Y(n)$.

The author analyses the stationary case as well as the situation for any initial distribution of $X(n)$ and any function f. The final two sections of the paper deal with the finite space and abstract state space.

(W. R. Buckland)

714

CONOLLY, B. W. **10.4** (0.4)

The busy period in relation to the single-server queueing system with general independent arrivals and Erlangian service-time—*In English*

J. R. Statist. Soc. B (1960) **22**, 89-96 (9 references)

The author considers the case where, instead of arriving singly, the arrivals take place in batches of a fixed size k. The time intervals separating the arrivals of successive batches are taken to be independently and arbitrarily distributed. The service time of each individual is assumed to be negative exponentially distributed. The batches are dealt with on a first come first serve basis, though the order of service of individuals comprising a batch is indifferent. This system simulates the queue which is given in Kendall's notation as $GI/E_k/1$.

By focussing attention on instants of arrival of batches, the writer set up a family of integro-difference equations, which are reduced to pure difference equations by the use of Laplace transforms. Attention is then restricted to a particular group of inter-arrival densities which will give simple roots for a particular equation, and the solution of the difference equations found.

The specific application to the busy period (when the server is fully occupied) is then made. The principal result is a formula for the generating function of the Laplace transforms with respect to time of the joint probability and probability density $p_m(t)$ that the busy period lasts for time t, and that in m groups are dealt with altogether.

Finally, two special cases are considered: first with $k = 1$ and general independence input and, secondly, the case of single random arrivals receiving service of random duration.

(C. Burrows)

715

COX, D. R. (Birkbeck College, London) **10.0** (1.8)

A renewal problem with bulk ordering of components—*In English*

J. R. Statist. Soc. B (1959) **21**, 180-189 (5 references, 3 tables)

Consider a system in which k similar components are in use at a time, and suppose that a batch of n components is obtained. When $n-k+1$ failures have occurred, so that only $k-1$ unbroken components remain, the life of the batch is considered to be ended. That is, the k components in use at one time must be from the same batch.

One example is connected with women's stockings; here $k = 2$ and it is assumed that any number, n, of identical stockings can be bought together and any two worn as a pair, but that when only one of the batch remains fit for use, it cannot be worn with stockings from a new batch and is scrapped.

If T is the life of the batch, a random variable, a simple index of efficiency is $I = [k\mathscr{E}(T)]/[n\mu]$, where μ is the mean life-time of a single component. The dependence of I is studied on

(i) the form of the distribution of life-time;

(ii) the values of n and k;

(iii) the strategy of use.

In connection with (iii) two strategies are considered, that of equalised wear and that of replacement on failure. In the first the components are used in turn for short periods, in such a way that at any time each component that has not failed has been in use the same total time. In the second, components are changed only on failure. It is assumed that the life-time of a component is independent of its strategy of use.

When the distribution of life-time is exponential, I is independent of strategy of use and is equal to $(n-k+1)/n$. In general, however, the value of I for the first strategy depends on properties of order statistics, and that for the second strategy can be calculated from renewal theory. Numerical results are given for a number of types of distribution of life-time.

(D. R. Cox)

716

DOBRUSCHIN, R. L. (Moscow University) **10.5** (–.–)

Transmission of information in channels with feedback—*In Russian*

Teor. Veroyat. Primen. (1958) **3**, 395-412 (9 references)

In this paper the author proves that the use of feedback does not increase the capacity of channels without memory. He also considers some simple channels with memory and compares their capacities when feedback is and is not used.

(R. L. Dobruschin)

DYNKIN, E. B. **10.1** (–.–)

One-dimensional continuous strong Markoff processes—*In Russian*

Teor. Veroyat. Primen. (1959) **4**, 3-54 (15 references)

One dimensional temporally-homogeneous strong Markoff processes with continuous path functions are considered. No regularity conditions are assumed and infinitesimal operators for all these processes are calculated. These calculations are based on a preliminary analysis of the local behaviour of path functions. The general results of sections 1-3 (classification, properties of continuous (Feller) processes, infinitesimal operators) are used in sections 4-5 which afford a more detailed investigation of the processes in intervals of their regularity.

Some results have already been published in 1955 without proofs in another paper by Dynkin [*Dokl. Akad. Nauk, SSSR,* **105**, 405-408].

(B. V. Gnedenko)

FELLER, W. (Princeton University) **10.1** (–.–)

Non-Markoffian processes with the semigroup property—*In English*

Ann. Math. Statist. (1959) **30**, 1252-1253 (1 reference)

The author shows that, for $N \geqq 3$ states, there exist non-Markoffian processes for which the n-step transition probabilities satisfy the Chapman-Kolmogoroff, or semi-group, relation, $\mathbf{P}^{n+m} = \mathbf{P}^n\mathbf{P}^m$, where $\mathbf{P}$ is the matrix of single-step transition probabilities. All elements of $\mathbf{P} = N^{-1}$.

(R. L. Anderson)

FINCH, P. D. (Research Tech. Unit, London School of Economics) **10.1** (2.5)

The output process of the queueing system M/G/1—*In English*

J. R. Statist. Soc. B (1959) **21**, 375-380 (4 references)

In this paper, the author studies a single server queue with Poisson input and a service time distribution with a finite mean and zero probability of instant service. This implies that the traffic intensity is less than unity when the queue is infinite, that is to say there is no baulking. If the queue length exceeds N, $N \geqq 1$, then there will be baulking.

In the development of this paper the author makes the assumption that the service time is a continuous variable with a differential density function.

The joint probability density function of the rth inter-departure time, l, and the number n, left in the queue after the departure is expressed as a sum of a finite number of values of the probability density function of n. The existence of the limit as r tends to infinity of the joint probability density function of the l and n then follows from the probability density function of n. If, in this limit, it is assumed that l and n are independent then an examination of the case for $n = 0$ shows that the service time must be exponentially distributed and that N is infinite.

Similar methods show that the same result follows from the assumptions that two successive interdeparture times tend to be independently distributed.

(D. E. Barton)

GAVER, D. P. (Westinghouse Research Laboratories, Pittsburgh) **10.4** (–.–)

Imbedded Markoff chain analysis of a waiting-line process in continuous time—*In English*

Ann. Math. Statist. (1959 **30**, 698-720 (14 references)

This paper studies single-server queueing systems with bunched arrivals and general independent service times. It is assumed that customers arrive according to a stationary compound process, and are served in order of arrival. The resulting process is analysed in continuous time by an adaptation of the method of the imbedded Markoff chain and by renewal theory. Busy period phenomena are discussed by the author and the joint distribution of busy period duration together with the number of customer departures in that duration is found explicitly: some properties of busy periods with " instantaneous defections " are derived. The transition probabilities for the number of customers in the system are represented by generating functions and Laplace-Stieltjes transforms. Later the ergodic properties of the process are discussed, and " long-run " distributions are found for both the number of customers in the system, and the waiting times.

(D. P. Gaver)

HAIGHT, F. A. (University of California, Los Angeles) **10.4** (–.–)

Overflow at a traffic light—*In English*

Biometrika (1959) **46**, 420-424 (5 references, 2 tables)

In this paper the author considers Poisson arrivals at an intersection controlled by a traffic light. During the green phase, the queue which has accumulated is discharged with constant time-separation between vehicles: if the queue becomes empty, any further arrivals pass through without obstruction and possibly with smaller time-separation than that mentioned above. Thus, it is only necessary to consider the input occurring while the queue is not empty.

The probability is calculated that there are z in the queue at the beginning of a red phase, given that there were x in the queue at the beginning of the previous green phase. The result is simple if either z is zero, or if x is greater than the maximum possible number of vehicles that can be discharged during a green phase. In the remaining cases, a recurrence relation can be set up, and the required probabilities can be written explicitly in terms of certain numerical coefficients. A recurrence relation is given for these and they are tabulated up to $z = x = 7$.

Assuming equilibrium, relations are given enumerating the probabilities of the numbers of vehicles in the queue at the commencement of adjusted red and green phases.

A semi-actuated system is described, in which the lights are governed by the traffic in a side street. The recurrence found previously can easily be generalised to cover this case; and if there is no maximum to the length of the red phase on the main street, the problem can be solved completely.

(C. L. Mallows)

HANANIA, MARY I. (University of California, Berkeley) **10.1** (5.8)

Two-alternative learning situations with partial reinforcement—*In English*

Psychometrika (1960) **25**, 77-90 (5 references, 2 tables)

A model related to the basic Bush-Mosteller learning model, but slightly more general in nature, is presented. The model treats the problem where the comparative effects of reward and non-reward on learning are considered in connection with a two-alternative learning situation. In the statistical model proposed, the question of whether reward and non-reward are equivalent in their effects on learning reduces to testing a composite hypothesis on a multivariate probability distribution. A lengthy and involved derivation of the model is given and an asymptotic test of the equivalence of reward and non-reward is described. The use of this test on an empirical problem is given.

(R. E. Stoltz)

HENN, R. (Handelschochschule, St Gallen, Switzerland) **10.1** (1.8)

Markoff chains in economic processes—*In German*

Metrika (1960) **3**, 61-73 (8 references, 6 tables, 3 figures)

This paper shows how to use models taken from the theory of Markoff chains for the description of economic developments. The author explains the difference between simple and multiple Markoff chains, defines the transition matrix and gives a sufficient condition for its convergence against a limit matrix whose elements determine the time-independent probability distribution of the states of the system. The examples treat population movements, the development of national income, trade cycles and the speculation with marketable stocks.

(W. Eberl)

HOWIE, A. J. & SHENTON, L. R. (Roy. Coll. Sci. & Tech., Glasgow and Manchester Coll. Sci. & Tech.) **10.4** (2.5)

The efficiency of automatic winding machines with constant patrolling times—*In English*

J. R. Statist. Soc. B (1959) **21**, 381-395 (4 references, 3 tables, 4 figures)

The problem is that of maximising the average efficiency of a machine which:

(i) stops when its total running time has reached a fixed value a

(ii) has accidental stops

(iii) is patrolled for servicing, that is to say if necessary at a fixed interval, P, which may be less than a.

The efficiency is defined as the ratio of the running time to the total time under consideration. It is assumed that the intervals between successive accidental stoppages have a negative exponential distribution.

A quantity i is defined by the fact that, if there are no accidental stoppages, the total running time of the machine will be greater than iP but less than $(i+1)P$. For a given value of i, a generating function is found of the probability that, due to accidental stoppages, s extra patrols will be necessary before the machine has been running for time a: the mean value of s is found. This leads directly to the average efficiency of the machine. A recurrence relation is derived for these probabilities and later explicit expressions are found in terms of Lageurre polynomials as well as a recurrence relation convenient for numerical calculation.

As a increases, it is shown that the distribution of the number of extra patrols is asymptotically normal. Finally, the relationship between the average efficiency and patrolling times are obtained for a given value of a and the accidental stoppage rate. Diagrams are given of the variation of efficiency with patrolling times.

(C. Burrows)

IKEDA, S. (Nihon University, Japan) **10.5** (–.–)

Continuity and characterisation of Shannon-Wiener information measure for continuous probability distribution—*In English*

Ann. Inst. Statist. Math., Tokyo (1959) **11**, 131-144 (5 references)

Following Shannon or Wiener, the author introduces the measure of information $H(p)$ for any probability density function p defined on some (σ-finite) measure space.

In the first place, the author shows that for the sequence of probability density functions $\{p_i\}$ and a probability density function p, if the essential supremum of the absolute values of the differences of p_i and p tends to zero and that of the ratio of p_i to p tends to one, then the sequence $H(p_i)$ converges to $H(p)$. This theorem is called the continuity theorem. If the total mass of the original measure space is finite, the condition about the ratios can be removed.

Secondly, the author discusses the characterisation of $H(p)$. Namely, when the probability density functions concerned are step functions, under some regularity conditions a functional of the probability density functions coincides with $H(p)$ with the possible exception of a positive constant factor if the continuity theorem holds concerning the sequence of the probability density functions. This theorem essentially depends on the fact that the step probability distribution functions are " dense " in the sense of the above mentioned convergence.

(M. Motoo)

KEILSON, J. & KOOHARIAN, A. (Sylvania Applied Res. Lab., Waltham, Mass.) **10.4** (0.4)
On time dependent queueing processes—*In English*
Ann. Math. Statist. (1960) **31**, 104-112 (5 references, 1 figure)

This paper considers the time-dependent version of the problem of a one server queue with Poisson arrival distribution and a general service time distribution. For discussion of the approach used here, as applied to the equilibrium case, see a paper by D. R. Cox [*Proc. Camb. Phil. Soc.* (1955) **51**, 433-441].

Let $W_m(x, t)$ be the probability density that at time t the queue length, excluding the present servee, is m, and that the elapsed service time is x. It is noted that $W_m(x, t)$ is defined independently of the queueing discipline, i.e., first come first served, random, etc. Differential equations governing $W_m(x, t)$ and the distribution, $E(t)$, of the completely vacant state are derived. These are solved in terms of the generating function

$$G(s, x, t) = \sum_{m=0}^{\infty} s^m W_m(x, t)$$

to give

$$G(s, x, t) = H_0(s, t-x)e^{-\lambda(1-s)x}e^{-N(x)}$$

where $N(x) = \int_0^x \eta(y)dy$, $\lambda =$ mean arrival rate, and $\eta(x)\Delta =$ first order probability that a service completion occurs in the time interval $(x, x+\Delta)$.

The problem is thereby reduced to an integro-differential equation in H_0 and the vacant state probability $E(t)$. This equation is analysed in detail for the special case where the system starts from the completely unoccupied state ($m = 0$, no service in progress). The Laplace transforms of H_0 and E are derived but not inverted. Further analysis is carried out for the case of exponential service time, yielding a more refined integral representation for $E(t)$. This integral is discussed further in a paper by P. M. Morse [*J. Operat. Res.* (1955) **3**, 255-261].

It is shown that in the equilibrium case, as t approaches infinity, $E(t)$ approaches $1-\lambda/\eta$, where $0 \leq \frac{\lambda}{\eta} < 1$ and η is the inverse of the mean service-time,

$$\eta = \left[\int_0^{\infty} xD(x)dx\right]^{-1},$$

and $D(x)$ is the service time density. The equilibrium solution for H_0 is also given. The authors state that, while both these are previously known results, they follow directly from the more general expressions derived in this paper.

(R. M. Durstine)

LESLIE, P. H. & GOWER, J. G. (Bur. Animal Pop., Oxford and Rothamsted Exp. Station) **10.9** (10.1)
The properties of a stochastic model for two competing species—*In English*
Biometrika (1959) **46**, 279

The authors notify a correction to equation (5.3) on page 327 of their paper already abstracted in this journal No. 135, **10.9**.

(W. R. Buckland)

LÉVY, P. (École Polytechnique, Paris) **10.0** (–.–)

Strict or almost strict Markoff processes—*In French*

Compos. Math., Groningen (1959) **14**, 172-193 (4 references)

This paper begins with the distinction between two kinds of definitions for a random function $X(t)$. It is weakly defined when the joint distribution of $X(t_1), X(t_2), \ldots, X(t_n)$ is given for every finite set of values t_ν of the parameter t. It is completely defined when $X(t)$ is a given function of t and of a set of independent random variables U_ν: the range of $X(t)$ is a non-enumerable set, different random functions may have the same weak definition; their complete definitions are necessarily different.

The random function $X(t)$ of the real parameter t is said to be Markoffian when, for every sequence of numbers t_n increasing with n, the sequence $X(t_n)$ is a simple Markoff chain. This property depends only on the weak definition of $X(t)$. It was necessary to have other definitions connected with the complete definition of $X(t)$. These were given by Dynkin & Youchkevitch, who defined the strong Markoff property. The author defines an almost strong Markoff property which has an intermediary character: it is more restrictive than the classical Markoff property, and less restrictive than the strong Markoff property.

A Markoff function $X(t)$ is weakly defined in (t_0, ∞) when the distribution of $X(t_0)$ and the transition probability is given. Starting from this data, the author gives a constructive method to obtain the complete definition of an almost strictly Markoff function. It is not always strongly Markoffian. It may happen that there exists, in the x-space E, critical points for which, when $X(t) = x$, the resolution depends on the previous values of the function. In this case, Ray defines an auxiliary space E^* and a correspondence between x and its points x^* with the following property: if x is a critical point, x^* is the logical product of x and a parameter that depends on the previous values; alternatively, $x^* = x$. Then, if $x = X(t)$, $x^* = X^*(t)$ is a strongly Markoffian function; and since, for every given t, one has almost surely $X^*(t) = X(t)$, it has the same weak definition as $X(t)$. The foregoing definition of the critical point is intuitive and not very precise.

In the last pages of his paper, the author tries to give a mathematical definition and proves the following theorem which is connected with the work of Chung: when the transition probability fulfills a suitable continuity condition, the function $X(t)$ has no critical point. Then the almost strong Markoff function that has been constructed above is a strong Markoff function, that has the given transition probability.

(P. Lévy)

729

MERCER, A. (Birkbeck Coll., London and Atomic Weapons Res., Aldermaston) **10.1** (1.5)

Some simple duration-dependent stochastic processes—*In English*

J. R. Statist. Soc. B (1959) **21**, 144-152 (4 references)

Two discrete states, in continuous time, are such that the transition probabilities from the occupied to the unoccupied state depend on the total time spent in one of the states. The joint probability that the process is in a given state and has spent a prescribed total time in one of the states, is derived. When the transition probabilities are powers of the total time, the limits of the following functions are obtained:

(i) the joint probability, defined above;
(ii) the probability of being in a given state;
(iii) the expected occupation time of a given state.

For constant transition probabilities, the mean and variance of the occupation time are given and the ergodic theorem of Darling & Kac is derived [*Trans. Amer. Math. Soc.* (1957) **84**, 444-458].

The distribution of the time taken for the occupation time of one of the states to reach a given value and expressions for the cumulants are obtained. Particular forms of the transition probabilities are given so that the limiting distribution is a chi-square distribution and a normal distribution.

(A. Mercer)

MERCER, A. (Birkbeck Coll., London and Atomic Weapons Res., Aldermaston) 10.4 (.)

A queueing problem in which the arrival times of the customers are scheduled—*In English*

J. R. Statist. Soc. B (1960) **22**, 108-113 (1 reference)

The queueing problem considered is that in which

(i) the customers are scheduled to arrive at equal time intervals but a customer may arrive at any time after the start of the interval during which he was scheduled to arrive, or may not arrive at all; the lateness distribution is perfectly general.

(ii) the total service-time of a customer is supposed to consist of a finite number of stages, such that the times spent in each of the stages are independent and identically distributed, and the probability of leaving a stage at any time depends on the time, u, since the current scheduling interval began.

A comparatively detailed mathematical argument is given, when there is only one server and the customer must either arrive in the scheduled interval, or in the next interval, or not at all. The method used is to consider the joint distribution of the queue length at time u, and whether or not each customer, who could arrive in the interval, has done so by time u. The probabilities are shown to satisfy a set of linked, first order, linear differential equations in u. These can be solved in turn, in terms of the probabilities at the beginning of the scheduling interval. Hence the probabilities at the beginning of all the scheduling intervals, and therefore at any time, may be found successively.

When the process is stationary, the probabilities at the end of a scheduling interval are equal to those at the beginning. Thus the probabilities at the start of a scheduling interval satisfy a system of recurrence relations, which are solved; the condition for the process to be stationary is also obtained.

The results for the equilibrium distribution are quoted in the two cases;

(*a*) more than one server;

(*b*) a customer may arrive during any interval after that for which he was scheduled.

(A. Mercer)

MESHALKIN, L. D. (Moscow University) **10.1** (1.5)

Limit theorem for Markoff chains with a finite number of states—*In Russian*

Teor. Veroyat. Primen. (1958) **3**, 361-385 (11 references)

The sequence of trials ($k = \overline{1-n}$; $n = 1, 2, \ldots$) is a uniform Markoff chain with a finite number of states $E_1, E_2, \ldots, E_s$. The matrix of transition probabilities is

$$\mathbf{P}(n) = || p_{uv}(n) ||_{u,\, v=1}^{s}$$

Let $\mu = \mu(n)$ denote the number of passages in the nth sequence of trials of the complete system on condition that the system is in state E at the initial moment. The author considers the limit distributions for a sequence of random variables $\alpha(\mu - n\theta)$; $\alpha = \alpha(n)$ and $\theta = \theta(n)$ are any functions of n. Theorems 1-5 give characteristic functions for possible limit distributions. Theorem 6 gives the main result of paper: if the limit distribution for $\alpha(\mu - n\theta)$ exists, then it does not differ from one of those found in theorems 1-5 by more than a linear transform.

(B. V. Gnedenko)

MIHOC, G. & FIRESCU, D. (Bucharest University) **10.2** (10.1)

Estimation functions for the parameters of a Markoff process with transition densities—*In Rumanian*

Analele Univ. " Parhon " Seria St. Naturii (1959) **22**, 9-16 (5 references)

The authors consider a real Markoff process whose transition densities depend upon a finite set of unknown parameters belonging to an open interval of the corresponding parameter space. Given a sample of volume n, a set of absolute correct estimation functions for the unknown parameters are considered. Under certain regularity conditions concerning the transition densities, necessary and sufficient conditions are given for the efficiency of the considered functions.

(R. Theodorescu)

733

MILLER, R. G., Jr. (Stanford University, California) **10.4** (0.4)

Priority queues—*In English*

Ann. Math. Statist. (1960) **31**, 86-103 (15 references)

In a priority queue the items arriving for service are divided into K classes, a type i item receiving priority over a type j item if $i<j$. Under a " head-of-the-line " (" first-come-first-served ") discipline the act of service is not interrupted by priorities; under a " pre-emptive " discipline, service on a type j item is actually stopped when a type i item ($i<j$) arrives. This latter discipline can operate under a " pre-emptive resume " policy (service on a pre-empted item later resumes at the point where it was interrupted) or a " pre-emptive repeat " policy (service on a pre-empted item must start over from the beginning).

In this paper the input processes are independent, and for type i items the input is a Poisson process with rate λ_i. The service time distribution F_{S_i} is general, subject only to $F_{S_i}(0+) = 0$ and $\mathscr{E}(S_i) < \infty$. There is one server.

The second section gives the stationary distributions of the number of items in the queue for $K = 2$. Earlier results obtained for exponential service times are extended to the general case (for the " head-of-the-line " priority queue) by the imbedded Markoff chain technique.

The next part of the paper deals with the distributions of waiting time, that is, the total time in the queue not counting service time. The Laplace-Stieltjes transform of the waiting-time distribution for the steady state is found ($K = 2$) for each type of item in the " head-of-the-line" discipline, and is expressed in terms of Laplace-Stieltjes transforms for busy periods which are characterised later in the paper. A similar expression is found for the type K item for cases where $K \geqq 2$: this section of the paper then characterises the distribution of waiting time (also time in service) for any type item in the " pre-emptive resume " queue, both for general time t and in equilibrium. It is stated that the corresponding problems for the " pre-emptive repeat " queue are for the most part unsolved.

Section 4 characterises the Laplace-Stieltjes transforms of the busy periods (as solutions of functional equations) for the queue with K classes and " head-of-the-line " (or pre-emptive resume) discipline. This is done for both the busy period of unspecified beginning and for the busy period beginning with a type i item. It is stated that no characterisation has been obtained for the pre-emptive repeat case.

The last part of the paper is concerned with the distribution of number of items served during a busy period in a " head-of-the-line " discipline (or pre-emptive resume) queue with K priority classes: generating functions are given as unique solutions of certain functional equations. Here, as in the earlier sections, the first and second moments of the distributions are found.

(R. Hooke)

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MOSTELLER, F. & TATSUOKA, M. (Harvard University and University of Hawaii) **10.1** (0.1)

Ultimate choice between two attractive goals: predictions from a model—*In English*

Psychometrika (1960) **25**, 1-17 (4 references, 3 tables)

This article presents a mathematical model for two-choice behaviour in situations where both choices are desirable. According to the model, one or the other choice is ultimately preferred, and a functional question is given for the fraction of the population ultimately preferring a given choice. The solution depends upon the learning rates and upon the initial probabilities of the choices. There is a numerical problem that has turned out to be rather troublesome. The authors sketch various solutions that have been tried. These procedures include iterative solutions, approximation by simultaneous equations, and approximation by differential equation. A comparison of the results obtained by the use of these methods is given.

The authors conclude that among the various approximate methods which can be easily carried out with desk calculators, the differential equation method yields results in closest agreement with those obtained by the simultaneous equations using twenty-one grid points and third-difference interpolation. An application of the model to a T-maze experiment with paradise fish is given. The learning model used by the authors is essentially one discussed by Bush & Mosteller.

The type of experimental problem to which the model may be applicable is one of the sort where an animal must choose between the right hand or left hand side in a T-maze. A more general case would be one where R and L are intended to stand for a general pair of attractive objects, mutually exclusive and exhaustive, which lead to attractive goals. On a given trial the probability of choosing R is p, and that of choosing L is $1-p$, where as usual $0 \leq p \leq 1$. If R is chosen, then the probability of choosing R the next time is increased to $\alpha_1 p+1-\alpha_1$, but if L is chosen the probability of choosing R the next time is reduced to $\alpha_2 p$, where $0 \leq \alpha_1 \leq 1$, $0 \leq \alpha_2 \leq 1$. Hence the transition rules: If p_n is the probability of R on trial n, then the probability of R on the next trial is

$$p_{n+1} = \begin{cases} \alpha_1 p_n + 1 - \alpha_1 & \text{if } R \text{ chosen on trial } n \\ \alpha_2 p_n & \text{if } L \text{ chosen on trial } n \end{cases}$$

From this one may derive the basic functional equation for the simple approach-approach problem:

$$f(p) = pf(\alpha_1 p + 1 - \alpha_1) + (1-p)f(\alpha_2 p).$$

The boundary conditions are $f(0) = 0$ and $f(1) = 1$.

(R. E. Stoltz)

MOY, S.-T. C. (Wayne State University, Detroit) **10.1** (1.5)

Successive recurrence times in a stationary process—*In English*

Ann. Math. Statist. (1959) **30**, 1254-1257 (2 references)

This note is concerned with the average time for successive recurrence times. Given that the first observation in a process is in state B, it is shown that successive recurrence times are stationary. A previous theorem by Kac [*Bull. Amer. Math. Soc.* (1947) **53**, 1002-1010] on the expectation of the first recurrence time is generalised to the kth recurrence time.

(R. L. Anderson)

MÜNZNER, H. (Freie Universität, Berlin) 10.5 (-.-)

On some basic concepts of information theory—*In German*

Blä. Dtsch. Ges. Versich.-math. (1960) **4**, 249-258.

The purpose of this note is to give an introduction to the fundamental conceptions of information theory. It starts with the definition of a message and the information content of a message or a symbol from a source. A discussion on the properties of the entropy of a random variable and a stationary Markoff-chain is followed by some remarks on the entropy of languages and the redundancy of a language.

The concepts of the capacity of a channel which is fundamental in the theory is illustrated by means of the simple example of a channel the input of which consists of a finite number of letters, each arising with a certain probability. The output consists of another finite group of symbols and the transition-probabilities are given. The capacity is then defined as the maximum of the mean information which can be transmitted by the channel, where the maximum is taken over all probability distributions the input alphabet can possibly have. The main theorem of Shannon is formulated and its consequences are explained. Only the most simple ideas of information theory are contained in this article, but the style allows the reader to deduce more than is written down explicitly.

(K. Prachar)

NASR, S. K. (Faculty of Science, Math. Dept., Alexandria) 10.1 (-.-)

A note on a noise process—*In English*

Metrika (1960) **3**, 46-52 (7 references)

The author considers the system of stochastic differential equations $\dot{x}(t) = \mathbf{A} \,.\, x(t) + \boldsymbol{\varepsilon}(t)$ where A is a square matrix with constant coefficients and $\boldsymbol{\varepsilon}(t)$ a vector of continuous random functions. He shows that it is equivalent to the matrix equation

$$\exp(-\mathbf{A}t)x(t) = \exp(-\mathbf{A}t_0)x(t_0) + \int_0^t \exp(-\mathbf{A}\tau)\boldsymbol{\varepsilon}(\tau)d\tau$$

which implies $x(t+1) = \exp(\mathbf{A})x(t) + \mathbf{h}(t)$ with a certain vector of random functions $\mathbf{h}(t)$, and he discusses the case that **A** is similar to a diagonal matrix. The paper concludes with some remarks concerning a method of estimating the elements of **A** due to Wald & Mann [*Econometrica* (1943) **11**, 173-220].

(K. Krickeberg)

OGAWARA, M. (Tokyo Women's College) **10.0** (10.2)
Time series analysis and stochastic prediction (I) & (II)—*In English*
Bull. Math. Statist. (1958-59) **8**, 8-53, and 55-72 (36 references, 1 table)

This is a combined review paper summing up the results of former papers (listed in the references of the present paper) by the same author. The content of this review paper is mainly concerned with statistical inference, under the assumption of normality, about stationary processes of the so-called finite parameter schemes such as autoregressive and moving average processes.

(H. Akaike)

PRABHU, N. U. (Karnatak University, Dharwar, India) **10.4** (2.5)
Application of generating functions to a problem in finite dam theory—*In English*
J. Aust. Math. Soc. (1959) **1**, 116-120 (4 refs.)

In this paper, the author applies Bailey's method for probability generating functions [*J. R. Statist. Soc.* B (1954) **16**, 80-87] to the derivation of stationary distributions for the storage Z_t at discrete times $t = 0, 1, 2, \ldots$, of a dam of finite capacity K units. Independently and identically distributed inputs $X_t (= 0, 1, 2, \ldots)$ flow into the dam in the time intervals $(t, t+1)$, and the dam is subject to an overflow Z_t+X_t-K if $Z_t+X_t>K$ in $(t, t+1)$, while a release of 1 or 0 units is made at $t+1$ depending on whether $Z_t+X_t \geqq 1$, or $Z_t+X_t = 0$.

For any geometric input distribution, it is shown that the stationary distribution of the storage is of the truncated geometric type with a modified first term. When the input is a particular negative binomial with probability generating function $G(z) = (1-b)^2(1-bz)^{-2}$ $(0<b<1)$, the stationary distribution is shown to be a linear combination of two truncated geometric distributions of the previous type. The analogous result for a general negative binomial input distribution is given by the author in a paper [*Ann. Math. Statist.* (1958) **29**, 1234-1243] where an alternative method of derivation is used.

(J. Gani)

Editorial Note: see abstract No. 310, **10.4.**

PRABHU, N. U. (Karnatak University, India) **10.4** (2.5)

Some results for the queue with Poisson arrivals—*In English*

J. R. Statist. Soc. B (1960) **22**, 104-107 (8 references)

This paper is concerned with the queueing system in which there is only one server, the customers arrive at random and are served in order of arrival. The service-time distribution is assumed to have a general form.

By considering the waiting time of a customer who arrives at an instant t_0 and the interval T before the waiting time reduces to zero for the first time, a joint distribution is found of T and N, where N is the number of " new " customers served during the interval (t_0, t_0+T). That is to say, excluding the customers standing in the queue or being served at t_0. From this the distribution of T is found.

Explicit expressions are then deduced for the distributions of the length of a busy period (when the server is fully occupied), the number of customers served during a busy period and the time taken for a queue of given length to disappear. Specific results are given for constant service-time and a negative exponential service-time distribution.

(C. Burrows)

741

PRÉKOPA, A. (Eötvös University, Budapest) **10.8** (1.4)

On secondary processes generated by random point distributions—*In English*

Ann. Univ. Sci. Budapest., Sect. Math. (1959) 2, 139-146 (3 references)

The author generalises his previous results concerning secondary processes [Prékopa, " On secondary processes generated by a random point distribution of Poisson type," *Ann. Univ. Sci. Budapest.*, Sect. Math. (1958) **1**, 153-170]. In the present paper the process which is generating the secondary process is of mixed-Poisson type. A random point distribution $\xi(A)$ is reckoned to be of mixed-Poisson type if there is a random variable λ, a σ-finite non-atomic measure $\gamma(A, \lambda)$ such that

$$P\{\xi(A_1) = k_1, \ldots, \xi(A_n) = k_n/\lambda\} = \prod_{i=1}^{n} \frac{\gamma^{k_i}(A_i, \lambda)}{k_i!} e^{-\gamma(A_i, \lambda)}$$

It is shown that if the random-point distribution $\eta(D)$ is generated by the random-point distribution of a mixed-Poisson type with the random-parameter measure $\gamma(A, \lambda)$ then

$$P\{\eta(D) = n\} = \int_0^\infty \frac{[\nu^*(D, \lambda]^n}{n!} e^{-\nu^*(D, \lambda)} d(\mu, \lambda) \ (n = 0, 1, 2, \ldots)$$

where ν^* is the extended measure of

$$\nu(D, \lambda) = \int_A \mu(C, \lambda, t)\gamma(dt, \lambda) \quad D = A \times C$$

$\mu(\lambda)$ is the probability distribution of the random variable λ and

$$\mu(C, \lambda, t) = P\{\eta(D) = 1/(\lambda, t)\} \quad t \in A.$$

In the second part of this paper some examples are given.

(P. Révész)

742

SEVASTIANOFF, B. A. **10.1** (2.5)

Transient phenomena in branching stochastic processes—*In Russian*

Teor. Veroyat. Primen. (1959) **4**, 121-135

The following scheme for generating particles is considered: each particle existing at a given time splits up into k particles with the probability $\delta_{k1}+p_k\Delta t+0(\Delta t)$, $\delta_{11}=1$, $\delta_{k1}=0$ for $k\neq 1$, in time $\Delta t\to 0$ independent of its age and origin, as well as the history of the other particles. We define the probability-generating function by $f(x)=\sum_{k=0}^{\infty} p_k x^k$ and denote factorial moments by $a=f'(1)$, $b=f''(x)$, $c=f'''(1)$. Let $K(b_0, c_0)$ be a class of $f(x)$ with $f''(1)\geqq b_0>0$ and $f'''(1)\leqq c_0<\infty$. Let μ_t be the number of particles at time t. We introduce the function

$$q(t, a)=\begin{cases}\dfrac{e^{at}}{1+\dfrac{b}{2a}(e^{at}-1)} & \text{for } a\neq 0,\\[2ex] \dfrac{1}{1+\dfrac{b}{2}t} & \text{for } a=0.\end{cases}$$

The following asymptotic formula for $t\to\infty$, $a\to 0$ holds true uniformly for all $f(x)\in K(b_0, c_0)$:

$$1-P\{\mu_t=0\mid_{\mu_0=1}\}\sim q(t, a).$$

The probability distribution

$$S_t(y)=P\left\{\frac{\mu_t}{\mu\{\mu_t\mid_{\mu_t>0}}<y\mid_{\mu_t>\mu_0=1}\right\}$$

converges to an exponential distribution as $t\to\infty$, $a\to 0$ uniformly for all $f(x)\in K(b_0, c_0)$. The asymptotic behaviour of $S_t(y)$ for $\mu_0=n$ and $n\to\infty$, $t\to\infty$, $a\to 0$ is investigated in section 3.

(B. Sevastianoff)

743

SMITH, W. L. (University of North Carolina, Chapel Hill) **10.1** (1.5)

On the cumulants of renewal processes—*In English*

Biometrika (1959) **46**, 502

The author notifies three corrections in pages 2, 18 and 19 of his paper already abstracted in this journal No. 315, **10.1**.

(W. R. Buckland)

744

VOLKOFF, I. S. 10.1 (–.–)

On the distribution of sums of random variables defined on a homogeneous Markoff chain with a finite number of states—*In Russian*

Teor. Veroyat. Primen. (1958) **3**, 413-429 (7 references)

Local and integral limit theorems are established for a non-periodical case. The results are given in form of asymptotic expansions taking into account various possible values of the sums under consideration.

(I. Volkoff)

WHITE, J. S. (Aero Div., Minneapolis-Honeywell Regulator Co., Minneapolis) 10.7 (1.5)

The limiting distribution of the serial correlation coefficient in the explosive case. II—*In English*

Ann. Math. Statist. (1959) **30**, 831-834 (2 references)

Given the stochastic difference equation, $x_t = \alpha x_{t-1} + u_t$, $t = 1, 2, \ldots, T$, where the u's are assumed to be normally and independently distributed with mean zero and variance σ^2, consider the estimators

$$\hat{\alpha} = \Sigma\, x_t x_{t-1} / \Sigma\, x_{t-}^2 \quad \text{and} \quad \hat{\sigma}^2 = \Sigma\, (x_t - \alpha x_{t-1})^2 / T.$$

This note shows that

$$W = (\hat{\alpha} - \alpha)(\Sigma\, x_{t-1}^2)^{1/2} / \hat{\sigma}$$

has a limiting normal distribution with zero mean and unit variance except, perhaps, when $|\alpha| = 1$. Mann & Wald had proved this for $|\alpha| < 1$. The present author extends it to the explosive case, $|\alpha| > 1$.

(R. L. Anderson)

WRIGHT, S. (University of Wisconsin, Madison) **10.9** (2.9)

On the number of self-incompatibility alleles maintained in equilibrium by a given mutation rate in a population of given size : a re-examination—*In English*

Biometrics (1960) **16**, 61-85 (13 references, 6 tables, 4 figures)

A re-examination has been conducted of an earlier paper by the author [*Genetics* (1939) **24**, 538-552] that dealt with the theoretical number of alleles maintained in equilibrium between a mutation rate and loss of alleles by sampling in a population of specified size. That research, deriving from the discovery by Emerson of self-incompatibility alleles in *Oenothera organensis*, has recently been treated by Fisher [*The Genetical Theory of Selection* (1958), Oxford: Clarendon Press]. Fisher has concluded from different numerical results that Wright's theory of recent reduction of the *Oenothera organensis* population, as well as partial isolation of colonies, is not correct. In answer to this criticism, four sources of possible discrepancies have been investigated in detail:

(i) *Rate of Change of Gene Frequency.* Different expressions for the mean rate of change of gene frequency have been compared for certain known sets of zygotic frequencies. It appears that Wright's expression for the mean is more accurate in distributions of the equilibrium type than Fisher's estimate, although it does not appear that the slight differences between these means are responsible for the discrepancies cited by Fisher.

(ii) *Sampling Variance of Rate of Change of Gene Frequency.* The conventional and exact formulae for the sampling variance differ only slightly for small allele frequencies. It does not seem that the earlier incorrect variance can be the cause of the large differences in the later results.

(iii) *Distribution of Gene Frequencies.* Apart from a term in the allele mutation rate, Fisher's probability density function for gene frequencies is the same as Wright's density.

(iv) *Number of Alleles and Rate of Turnover.* A more accurate factor is offered to give the total rate of loss per gene. Fisher, on the other hand, does not consider the problem presented by the partial isolation of the colonies of the species, nor does he balance his equation for equilibrium in the presence of high mutation rates. Fisher's calculation of the number of alleles is only approximate.

Wright's earlier theoretical distributions of self incompatibility alleles have been recalculated with different gene frequency change rates to effect greater comparability with Fisher's distributions. Only one difference of real importance obtains: Fisher's formula for the mean rate of change of gene frequency gives a six-fold greater mutation rate. Fisher's two more recent distributions have also been

recalculated with various combinations of the newer expressions. In both cases, the graphs of the distributions plotted from different expressions for the parameters are generally equivalent, although in the first case of a population of $N = 1000$, a difference is present between the numbers of alleles, while in the second ($N = 10{,}000$), Fisher's estimate of the number of alleles lost per generation is greater by a factor of 63.

(D. F. Morrison)

FEDERIGH, E. T. (Bendix Radio, Maryland)
Extended tables of the percentage points of Student's t distribution—*In English* **11.1** (3.1)
J. Amer. Statist. Ass. (1959) **54**, 683-688 (2 references, 1 table)

This paper fills a need in using Student's t distribution when the upper percentage points corresponding to very low tail areas as well as large numbers of degrees of freedom are required. A table of percentage points corresponding to the following probabilities is given: (25, 10, 5, 2·5) times 10^{-2}; (10, 5, 2·5) times 10^{-3}, 10^{-4}, 10^{-5}, 10^{-6}, 10^{-7}, respectively. The degrees of freedom extend through 1(1) 30(5) 60(10) 100, 200, 500, 1000, 2000, 10000. All the entries in the tables are correct to 3 decimal places.

The upper portion of the table (degrees of freedom ≤ 30) was obtained by integrating the density function of Student's t distribution directly, and applying finite series expressions developed by Dunnett. The lower portion of the table (degrees of freedom ≥ 50) was obtained from an asymptotic expansion using a method of Fisher. The middle part of the table was obtained by asymptotic interpolation.

These tables point out errors in the *Biometrika* tables for 2 and 3 degrees of freedom and $P = 0{\cdot}001$. They also point out errors for 3, 5 and 7 degrees of freedom and $P = 0{\cdot}0005$ in Fisher & Yates' *Statistical Tables for Biological, Medical and Agricultural Research.*

(J. Gurland)

749

KAPLAN, N. M. (RAND Corporation, Santa Monica, Cal.)
Some methodological notes on the deflation of construction—*In English* **11.8** (–.–)
J. Amer. Statist. Ass. (1959) **54**, 535-555 (6 tables)

This article contains a review of methods used in computing indexes of construction (or more precisely price indexes to be used in the deflation of construction). The author suggests that an index of " prices of intermediate products " is superior to indexes of input prices such as those in current use. An index of prices of intermediate or component parts becomes logical when, although the final products themselves may not be comparable, construction is viewed as a sum of relatively homogeneous components. Two methods of computing price indexes of such components are suggested.

(W. A. Fuller)

750

LOWE, J. R. (Ministry of Defence, London) **11.1** (2.3)

A table of the integral of the bivariate normal distribution over an offset circle—*In English*

J. R. Statist. Soc. B. (1960) **22**, 177-187 (3 references, 1 table)

A bivariate normal surface is supposed in which the two variables are uncorrelated, have different standard derivations σ_1, and σ_2, but both have zero means. A circle of radius a and centre (u, v) is considered and the value of the bivariate normal surface over the area of this circle has been calculated. The method of calculation is not stated.

The probability over the circle area is given for values of u/σ_1 and v/σ_2 for the ratios $\sigma_1/\sigma_2 = 1, 2, 4, 8$, and $a/\sigma_2 = 1, 2, 4, 8, 16, 32, 64$: three decimal places are given.

(Florence N. David)

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